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Improving simple explicit methods for unsteady open channel and river flow

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SUMMARY

A rigorous study of the explicit Lax–Friedrichs scheme for its application to one-dimensional shallow water flows is presented. The deficiencies of this method are identified and the way to overcome them are presented. It is compared to the explicit first order upwind scheme and to the explicit second order Lax–Wendroff scheme by means of the simulation of several test cases with exact solution. All three schemes in their best balanced version are applied to the simulation of a real river flood wave leading to very satisfactory results. Copyright © 2004 John Wiley & Sons, Ltd.

KEY WORDS: explicit methods; Lax–Friedrichs scheme; Lax–Wendroff scheme; river flow; one-dimensional shallow water flow

1. INTRODUCTION

Explicit numerical schemes are based on the Euler time integration rule. They use the value of the variables at a known time level t^n so that the unknown values at a new time level t^{n+1} depend only on them. This conceptual simplicity justifies the wide acceptance and application of explicit schemes to time dependent problems in Computational Fluid Dynamics. There is a disadvantage common to these methods; it is the limitation on the time step size imposed by the Courant–Friedrichs–Lewy [1] stability condition which can become highly restrictive. Their application to transient flows is nevertheless extensive and frequent and this work is devoted to the study of some of the best known explicit schemes.

The first explicit methods developed to solve transient flow equations were based on central differences. Among them, one of the best known is the Lax–Friedrichs scheme [2]. It is a simple and robust scheme which however relies on an artificial global viscosity coefficient that affects the numerical solutions. It can become excessively diffusive and inaccurate when used to simulate steady state solutions in the context of shallow water flows. In an attempt to overcome these limitations, Lax–Wendroff [3] proposed a second order in space and time scheme, based on the series development of the differential equations up to second order in

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time. Their method is more accurate but, on the other hand, can generate spurious numerical oscillations near sharp gradients. The Lax–Wendroff scheme requires the estimation and use of the flux Jacobian. For some problems this can be cumbersome and traditionally has represented a disadvantage.

From a different approach, upwind schemes are based on non-centered differences biased towards the physical region of dependence of the physical equation. They were first proposed by Courant *et al.* [4] for hyperbolic equations in characteristic form and represented a milestone in shock propagation problems. In a coupled system of non-linear equations such as the shallow water equations, it may be difficult to identify correctly the region of influence associated to every propagation speed and, at the same time, to preserve the conservative character of the scheme. This is the reason why upwind schemes tend to be more complex than central schemes. One of the first conservative forms proposed were the '*flux vector splitting*' [5, 6], which do not have general validity for systems of non-linear equations.

Godunov [7] was the first to introduce the idea of solving non-linear systems as a series of local Riemann problems with exact solution. A number of approximate Riemann solvers came later on in an attempt to simplify Godunov's idea. Among them, Roe's first order method deserves special mention [8]. Further approaches incorporating more nodes to the discretization of the spatial derivatives led to higher order upwind methods with excellent properties.

Both upwind and central schemes solve incorrectly the flow accelerations with transition from sub- to supercritical flow leading to unphysical solutions. Lax–Friedrichs scheme is the exception due to its artificial viscosity. Harten and Hyman [9] where the first to correct Roe's scheme in order to yield physically acceptable solutions by means of a suitable artificial viscosity.

Almost all the numerical schemes used in Computational Hydraulics derive from original developments in Gas Dynamics. One of the main difficulties in the derivation comes from the presence of topography effects under the form of source terms in the equations. They contain derivatives and can be dominant in some cases. Glaister [10] first proposed the upwind discretization of source terms in the context of upwind schemes. The improvement of earlier ideas has been investigated, mainly for steady state solutions by [11–13].

In this work, the basic properties of general one-dimensional schemes will be presented and analysed for the Lax–Friedrichs scheme, Lax–Wendroff scheme and the first order upwind scheme. An optimized Lax–Friedrichs scheme is proposed so that, depending on the choice of the artificial viscosity applied, it behaves as the first order upwind or the Lax–Wendroff scheme showing how a very simple and easy to use numerical method can improve its properties. Several cases with analytical solutions are presented including some with bed slope and friction effects. Finally all four schemes will be applied to a realistic river flow problem and their performance compared.

2. DIFFERENT FORMS OF THE GOVERNING EQUATIONS

Different forms of the one-dimensional shallow water or St. Venant equations can be written. They all start from a cross section average of the basic flow equations and the underlying hypothesis of hydrostatic pressure distribution p

$$p = p_{\mathrm{at}} + g(z_{\mathrm{s}} - z)$$

where p_{at} is the atmospheric pressure, z_{s} is the free surface level and z represents the vertical level of a generic point. From this assumption, the cross section averaged pressure term derivative along the main flow direction is

$$A\frac{\partial p}{\partial x}=\int_{z_{\mathrm{b}}}^{z_{\mathrm{s}}}\sigma(x,z)\frac{\partial}{\partial x}[p_{\mathrm{at}}+g(z_{\mathrm{s}}-z)]\,\mathrm{d}z=\int_{z_{\mathrm{b}}}^{z_{\mathrm{s}}}\sigma(x,z)g\frac{\partial z_{\mathrm{s}}}{\partial x}\,\mathrm{d}z=gA\frac{\partial z_{\mathrm{s}}}{\partial x}$$

where A is the cross sectional area, z_{b} is the bottom level and $\sigma(x,z)$ is the variable channel width. Assuming also adequate and convenient friction and turbulence models denoted by E, a first version of the cross sectional averaged differential equations is

$$\begin{aligned}\frac{\partial A}{\partial t}+\frac{\partial Q}{\partial x}&=0\\ \frac{\partial Q}{\partial t}+\frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right)&=E-gA\frac{\partial z_{\mathrm{s}}}{\partial x}\end{aligned}\tag{1}$$

in terms of the cross sectional area and discharge Q. This may be seen as a quasi-conservative form of the type

$$\frac{\partial\mathbf{u}(x,t)}{\partial t}+\frac{\mathrm{d}\mathbf{F}^{\mathrm{qc}}(x,\mathbf{u})}{\mathrm{d}x}=\mathbf{S}^{\mathrm{qc}}(x,\mathbf{u})\tag{2}$$

in which $\mathbf{u}$ is the conserved variable, $\mathbf{F}^{\mathrm{qc}}$ the flux and $\mathbf{S}^{\mathrm{qc}}$ the source term:

$$\mathbf{u}=\begin{pmatrix}A\\Q\end{pmatrix},\quad\mathbf{F}^{\mathrm{qc}}=\begin{pmatrix}Q\\\frac{Q^2}{A}\end{pmatrix},\quad\mathbf{S}^{\mathrm{qc}}=\begin{pmatrix}0\\E-gA\,\frac{\mathrm{d}z_{\mathrm{s}}}{\mathrm{d}x}\end{pmatrix}$$

but this flux, however, does not contain all the information relevant to propagation phenomena in open channel flows. A fully conservative form of the equations can be built from (1). The steps to follow start using I_1 as the following integral:

$$I_1=\int_{z_{\mathrm{b}}}^{z_{\mathrm{s}}}\sigma(x,z)(z_{\mathrm{s}}-z)\,\mathrm{d}z$$

and the property that

$$\frac{\mathrm{d}}{\mathrm{d}x}(gI_1)=g\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_0^h\sigma(h-z')\,\mathrm{d}z'\right]=g\left(A\frac{\mathrm{d}h}{\mathrm{d}x}+I_2\right)$$

$$\frac{\mathrm{d}z_{\mathrm{s}}}{\mathrm{d}x}=\frac{\mathrm{d}h}{\mathrm{d}x}+\frac{\mathrm{d}z_{\mathrm{b}}}{\mathrm{d}x}$$

with h the water depth and:

$$I_2=\int_0^h\frac{\mathrm{d}\sigma(x,z')}{\mathrm{d}x}(h-z')\,\mathrm{d}z'$$

representing the pressure forces exerted by the walls in the flow direction and the derivative $(\mathrm{d}\sigma(x,z')/\mathrm{d}x)$ defined over the variable $z'=z-z_{\mathrm{b}}$. The conservative form is therefore:

$$\frac{\partial\mathbf{u}(x,t)}{\partial t}+\frac{\mathrm{d}\mathbf{F}^{\mathrm{c}}(x,\mathbf{u})}{\mathrm{d}x}=\mathbf{S}^{\mathrm{c}}(X,\mathbf{u})\tag{3}$$

with the new flux and source terms:

$$\mathbf{u} = \begin{pmatrix} A \\ Q \end{pmatrix}, \quad \mathbf{F}^{\mathrm{c}} = \begin{pmatrix} Q \\ \frac{Q^2}{A} + gI_1 \end{pmatrix}, \quad \mathbf{S}^{\mathrm{c}} = \begin{pmatrix} 0 \\ E + g(I_2 - A\frac{\mathrm{d}z_{\mathrm{b}}}{\mathrm{d}x}) \end{pmatrix}$$

The flux in this formulation contains all the relevant physical terms.

It is worth noting here that a careful distinction must be made between total and partial spatial derivatives:

$$\frac{\mathrm{d}\mathbf{F}(x,\mathbf{u})}{\mathrm{d}x} = \left[\frac{\partial \mathbf{F}(x,\mathbf{u})}{\partial x}\right]_{\mathbf{u}=\mathrm{cte}} + \left[\frac{\partial \mathbf{F}(x,\mathbf{u})}{\partial \mathbf{u}}\right]_{x=\mathrm{cte}} \frac{\partial \mathbf{u}(x,t)}{\partial x} = \frac{\partial \mathbf{F}(x,\mathbf{u})}{\partial x} + \mathbf{J}(x,\mathbf{u})\frac{\partial \mathbf{u}(x,t)}{\partial x} \tag{4}$$

with $\mathbf{J} = \partial\mathbf{F}/\partial\mathbf{u}$ the flux Jacobian. Therefore, $(\mathrm{d}\mathbf{F}/\mathrm{d}x) \neq (\partial\mathbf{F}/\partial x)$ and it should be remembered that, at the discrete level, finite differences are always approximations of the total derivative. This is very important and has traditionally been neglected. From the conservative form (3), a new, non-conservative form can be derived using (4):

$$\frac{\partial \mathbf{u}(x,t)}{\partial t} + \mathbf{J}(x,\mathbf{u})\frac{\partial \mathbf{u}(x,t)}{\partial x} = \mathbf{S}^{\mathrm{nc}}(x,\mathbf{u}) \tag{5}$$

where

$$\mathbf{J} = \frac{\partial \mathbf{F}^{\mathrm{c}}}{\partial \mathbf{u}} = \begin{pmatrix} 0 & 1 \\ c^2 - u^2 & 2u \end{pmatrix}, \quad \mathbf{S}^{\mathrm{nc}} = \mathbf{S}^{\mathrm{c}} - \frac{\partial \mathbf{F}^{\mathrm{c}}}{\partial x} = \begin{pmatrix} 0 \\ E - gA\frac{\mathrm{d}z_{\mathrm{s}}}{\mathrm{d}x} + c^2\frac{\partial A}{\partial x} \end{pmatrix}$$

being $c = \sqrt{g(A/B)}$ the celerity of the small surface perturbations and B the channel top width.

The Jacobian in non conservative form (5) can be diagonalized and the equations in the system can be made independent. If $\mathbf{P}$ is the matrix diagonalizing the flux Jacobian $\mathbf{J}$, they verify

$$\mathbf{J} = \mathbf{P}\mathbf{\Lambda}\mathbf{P}^{-1}, \quad \mathbf{\Lambda} = \mathbf{P}^{-1}\mathbf{J}\mathbf{P}$$

with $\mathbf{\Lambda}$ the diagonal matrix made of the eigenvalues of $\mathbf{J}$.

For the shallow water system these matrices are:

$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}, \quad \mathbf{P}^{-1} = \frac{1}{\lambda_2 - \lambda_1}\begin{pmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{pmatrix}, \quad \mathbf{\Lambda} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{6}$$

with λ the eigenvalues of $\mathbf{J}$, $\lambda_1 = u + c$, $\lambda_2 = u - c$. Left product of (5) by matrix $\mathbf{P}^{-1}$ gives:

$$\mathbf{P}^{-1}\frac{\partial \mathbf{u}}{\partial t} + \mathbf{\Lambda}\mathbf{P}^{-1}\frac{\partial \mathbf{u}}{\partial x} = \mathbf{P}^{-1}\mathbf{S}^{\mathrm{nc}} \tag{7}$$

and allows the definition of the characteristic variables $\mathbf{w}$ holding the following relation at differential level:

$$\mathrm{d}\mathbf{w} = \mathbf{P}^{-1}\,\mathrm{d}\mathbf{u} \tag{8}$$

Substitution in (7) provides the characteristic form of the differential equations

$$\frac{\partial \mathbf{w}(x,t)}{\partial t} + \mathbf{\Lambda}(x,\mathbf{w})\frac{\partial \mathbf{w}(x,t)}{\partial x} = \mathbf{P}^{-1}(x,\mathbf{w})\mathbf{S}^{\mathrm{nc}}(x,\mathbf{w}) \tag{9}$$

3. TEST CASES WITH EXACT SOLUTION

The equations presented in the previous section do not have exact or analytical solution in general but only in a few particular cases. That is the reason why numerical methods have to be studied, adapted and carefully applied. As presented before, the equations can be cast under the form of conservation equations with source terms. The best form to start analysing the properties and performance of the numerical schemes is to use them to solve cases with exact solution. A few basic one-dimensional test cases of academic conservation laws with exact solution will be presented first: the linear advection equation and the inviscid Burgers equation. Going further the ideal dam break problem will be used as the purest advection example in the shallow water context. Then, the influence of the source terms will be analyzed by means of a still water equilibrium case and, finally, two steady channel flow problem as stated by MacDonald will help to validate the methods in presence of source terms [14].

3.1. Linear advection equation

The simplest conservation law is the linear advection equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \tag{10}$$

where u is the conserved/transported variable and a is the constant advection speed. All the differential forms previously identified in the shallow water system are identical in this trivial case, being the characteristic lines straight and parallel. Given initial conditions, the exact solution to this equation is

$$u(x,t) = u(x - at, 0) \tag{11}$$

It is usual to study the advection of initial square waves like

$$u(x,0) = \begin{cases} u_{\max}, & \forall x \in [x_0 - \frac{w}{2}, x_0 + \frac{w}{2}] \\ u_{\min}, & \forall x \notin [x_0 - \frac{w}{2}, x_0 + \frac{w}{2}] \end{cases} \tag{12}$$

where w represents the square width, due to the challenge associated to the discontinuities. Another frequent case is that of a gaussian profile like

$$u(x,0) = u_{\min} + (u_{\max} - u_{\min}) \exp\left[-\left(2\frac{x - x_0}{w}\right)^2\right] \tag{13}$$

We shall use a domain $[0, 100]$, and values $x_0 = 20, w = 20, u_{\min} = 0.2$ and $u_{\max} = 0.8$, to compare performances at a time $t = 60$. The units are not specified as we are not dealing with any particular physical magnitude.

3.2. Inviscid Burgers equation

The inviscid Burgers equation is a non-linear advection equation closely related to the inviscid flow equations. It represents an intermediate step between the simplicity of the linear advection and the complexity of the non linear systems. In conservative and non-conservative form it is written as

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=0 \tag{14}$$

The analytical solution to this equation can be easily derived from its characteristic form only for spatially increasing initial conditions since, in this case, the straight characteristic lines diverge. When the initial conditions are spatially decreasing functions, the characteristic lines tend to converge and eventually overtake each other. This corresponds to discontinuous solutions. A simetrical transcritical discontinuous initial condition like

$$u(x,0)=\begin{cases} u_0, & \forall x\leqslant 0\\ -u_0, & \forall x>0\end{cases}$$

will not be transported since

$$\frac{\partial u}{\partial t}=-\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)=0, \quad \forall x$$

and, in general, the evolution of a decreasing jump like

$$u(x,0)=\begin{cases} u_{\max}, & \forall x\leqslant 0\\ u_{\min}, & \forall x>0\end{cases}$$

is dictated by a jump (shock) speed

$$U=\frac{u_{\max}+u_{\min}}{2} \tag{15}$$

The inviscid Burgers equation is useful to calibrate the behaviour of the numerical schemes both in propagating non-linear discontinuities and in coping with sign changes in the function (transcritical points). The following initial conditions consisting of a square shape with two transcritical jumps will be used as test case:

$$u(x,0)=\begin{cases} -u_1, & \forall x\in(-\infty,x_1)\cup(x_2,\infty)\\ u_2, & \forall x\in[x_1,x_2]\end{cases}$$

The corresponding analytical solution is

$$t_c=2\frac{x_2-x_1}{u_2+u_1}$$

$$t\leqslant t_c\Rightarrow u(x,t)=\begin{cases} -u_1, & x\in(-\infty,x_1-u_1t)\cup(x_2+\frac{u_2-u_1}{2}t,\infty)\\ \dfrac{x-x_1}{t}, & x\in[x_1-u_1t,\,x_2+u_2t)\\ u_2, & x\in[x_2+u_2t,\,x_2+\frac{u_2-u_1}{2}t]\end{cases}$$

$$t > t_c \Rightarrow u(x,t) = \begin{cases} -u_1, & x \in (-\infty, x_1 - u_1 t) \cup (x_1 + (u_2 + u_1)\sqrt{tt_c}, \infty) \\ \dfrac{x - x_1}{t}, & x \in [x_1 - u_1 t, x_1 + (u_2 + u_1)\sqrt{tt_c}] \end{cases}$$

The domain used will be [0, 100], with $x_1 = 32.5$, $x_2 = 77.5$, $u_1 = 1$, $u_2 = 2$ for a time $t = 20$.

3.3. Ideal dam break

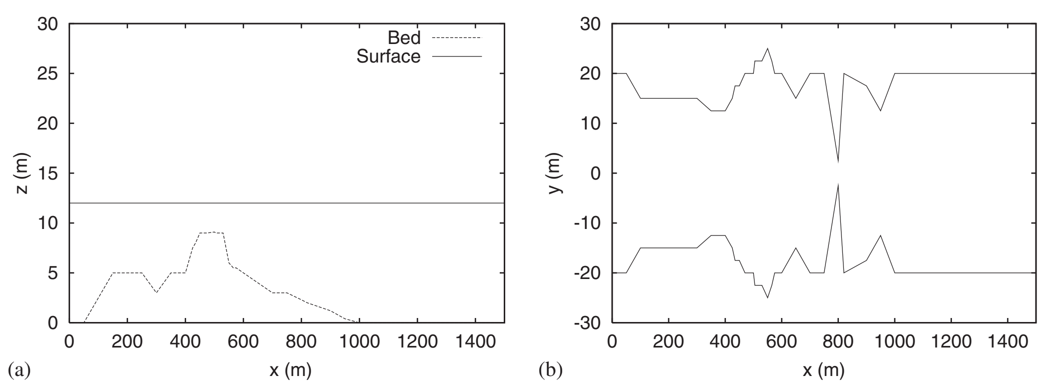
The dam break problem is one of the most classical unsteady problems with discontinuous analytical solution. When discontinuous initial conditions are assumed in a prismatic, flat and frictionless channel, the theory of characteristics and that of shock waves together lead to the solution [15]. This solution consists of a depression wave linked to a shock wave by an acceleration branch. This branch can go through a critical section depending only on the initial water level discontinuity. An initial discontinuity in the free surface of 1:0.1 in a 200 m long channel will be used as test case and the solution analyzed at $t = 20$ s to avoid interaction with the boundaries.

3.4. Hydrostatic equilibrium

Still water situations in presence of variable bed and channel shape are a challenging problem for advection schemes. In this case, the equations in quasi-conservative form (1) reduce to

$$\frac{\partial z_s}{\partial x} = 0$$

that is, the free surface level is uniform. Advection schemes are not always able to keep the static equilibrium at the discrete level. A test case proposed by Goutal & Maurel [16] has been selected. It is a channel rectangular in cross section with variable width and bed level as Figure 1 shows. A Manning coefficient $n = 0.01$ is assumed. The evolution in time of an initial uniform 12 m free surface level of still water will be studied during 200 s in a 150 cell grid.



(a) (b)

Figure 1. Goutal & Maurel channel: (a) initial water surface profile and bed level variation, (b) plant view.

3.5. MacDonald's test case

When the shallow water equations are used to model hydraulic problems involving bed slope changes and bed friction, the system is no longer homogeneous and the source terms have to be taken into account. On the other hand, this renders more difficult and often impossible to find exact solutions for validation. MacDonald [14] proposed a set of test cases based on steady flow in channels of varying bed slope and/or breadth by calculating the analytical slope and breadth functions compatible with constant discharge conditions given an analytical water depth function. Among them, we have chosen two examples. The first (MacDonald-1) concentrates on the behaviour in case of subcritical and continuous water profile and is a rectangular channel 150 m long and defined by Figures 2(a) and 2(b). A second example (MacDonald-2) consists of a 650 m long trapezoidal channel with a bed variation given by a slope function of x as depicted in Figure 3(a) and 3(b). In both cases a roughness coefficient $n = 0.03$ and a constant discharge $Q = 20\,\mathrm{m}^3/\mathrm{s}$ are assumed. There are some points of transcritical flow. In both cases a 400 cell grid will be used.

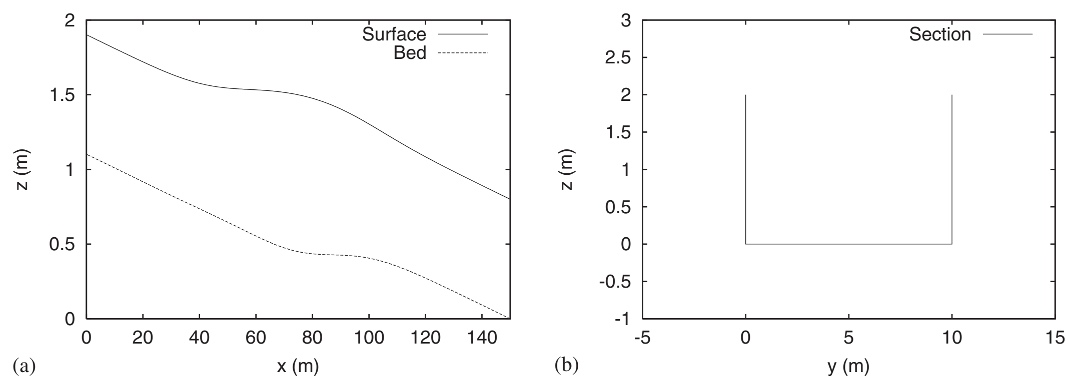


Figure 2. Analytical solution for the bed and depth profiles (a) and cross section (b) in the MacDonald-1 test case.

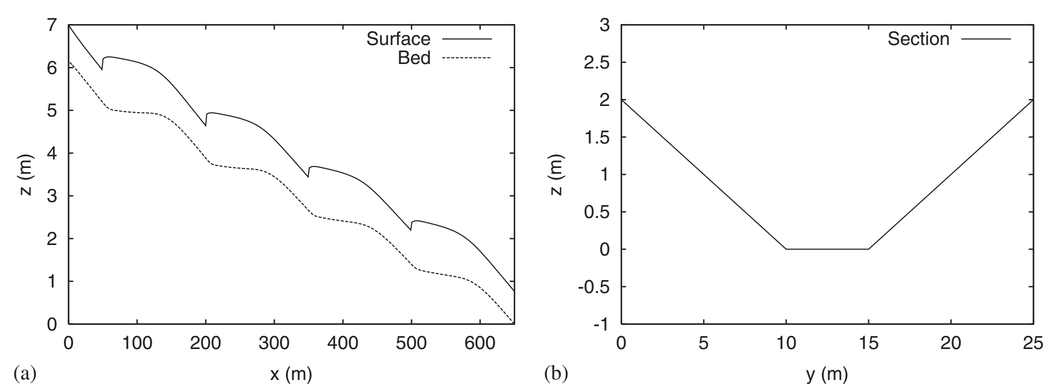


Figure 3. Analytical solution for the bed and depth profiles (a) and cross section (b) in the MacDonald-2 test case.

4. EXPLICIT SCHEMES

Three well known explicit numerical schemes will be considered here. Their basic properties regarding stability, monotonicity and conservation will be stated. For the sake of completeness and easy reading, definitions are provided in the appendix at the end of the manuscript.

4.1. Lax–Friedrichs scheme

The original scheme of Lax–Friedrichs [3] was proposed for equations and systems of conservation laws without source terms in the form

$$\Delta \mathbf{u}_i^n = \frac{1}{2}(\delta \mathbf{u}_{i+(1/2)}^n - \delta \mathbf{u}_{i-(1/2)}^n) - \frac{\Delta t}{2}\left[\left(\frac{\delta \mathbf{F}}{\delta x}\right)_{i-(1/2)}^n + \left(\frac{\delta \mathbf{F}}{\delta x}\right)_{i+(1/2)}^n\right] \tag{16}$$

A different version was later proposed [17]

$$\Delta \mathbf{u}_i^n = \frac{1-\alpha}{2}(\delta \mathbf{u}_{i+(1/2)}^n - \delta \mathbf{u}_{i-(1/2)}^n) - \frac{\Delta t}{2}\left[\left(\frac{\delta \mathbf{F}}{\delta x}\right)_{i-(1/2)}^n + \left(\frac{\delta \mathbf{F}}{\delta x}\right)_{i+(1/2)}^n\right] \tag{17}$$

where α is a weighting parameter related to artificial viscosity. The extension of Lax–Friedrichs scheme to equations with source terms is open to different possibilities. The simplest is a nodal or pointwise discretization as follows:

$$\Delta \mathbf{u}_i^n = \frac{v}{2}(\delta \mathbf{u}_{i+(1/2)}^n - \delta \mathbf{u}_{i-(1/2)}^n) + \Delta t \mathbf{S}_i^n - \frac{\Delta t}{2}\left[\left(\frac{\delta \mathbf{F}}{\delta x}\right)_{i-(1/2)}^n + \left(\frac{\delta \mathbf{F}}{\delta x}\right)_{i+(1/2)}^n\right]$$

with $v = 1 - \alpha$. Sometimes, in presence of important source terms like those appearing in practical river flow applications, an implicit or semi-implicit treatment is necessary:

$$\mathbf{S}_i^{n+\theta} = \theta \mathbf{S}_i^{n+1} + (1-\theta)\mathbf{S}_i^n \approx \mathbf{S}_i^{n+} + \theta \mathbf{K}_i^n \Delta \mathbf{u}_i^n \tag{18}$$

with $\mathbf{K} = \partial \mathbf{S}/\partial \mathbf{u}$ the Jacobian of the source terms, and:

$$(1-\theta \Delta t \mathbf{K}_i^n)\Delta \mathbf{u}_i^n = \frac{v}{2}(\delta \mathbf{u}_{i+(1/2)}^n - \delta \mathbf{u}_{i-(1/2)}^n) + \Delta t \mathbf{S}_i^n - \frac{\Delta t}{2}\left[\left(\frac{\delta \mathbf{F}}{\delta x}\right)_{i-(1/2)}^n + \left(\frac{\delta \mathbf{F}}{\delta x}\right)_{i+(1/2)}^n\right] \tag{19}$$

Another option is a non-centred discretization of the source term. In this case, and keeping pointwise the implicit part for simplicity, the Lax–Friedrichs scheme (LF) becomes

$$(1-\theta \Delta t \mathbf{K}_i^n)\Delta \mathbf{u}_i^n = \frac{v}{2}(\delta \mathbf{u}_{i+(1/2)}^n - \delta \mathbf{u}_{i-(1/2)}^n) + \frac{\Delta t}{2}(\mathbf{G}_{i-(1/2)}^n + \mathbf{G}_{i+(1/2)}^n) \tag{20}$$

where $\mathbf{G}$ can be defined in any of the equivalent forms, conservative (A7), quasi-conservative (A8) or non-conservative (A9).

On the other hand, being a three point scheme of type (Al5), its coefficients for the linear advection equation are:

$$A = B = 0, \quad C = -\frac{a\Delta t}{2\Delta x}, \quad D = \frac{v}{2}$$

and using (A18), it is controlled by the stability condition:

$$\left|a\frac{\Delta t}{\Delta x}\right|^2 \leqslant v \leqslant 1$$

When applied to a system of equations like the shallow water system in conservative form, it can be said that the Lax–Friedrichs scheme is stable and dissipative when:

$$\mathrm{CFL}^2 < v < 1 \tag{21}$$

where CFL is the Courant Friedrichs Lewy number [1] defined as

$$\mathrm{CFL} = \max\left|\lambda_k \frac{\Delta t}{\Delta x}\right| \tag{22}$$

with λ_k the eigenvalues of the flux Jacobian matrix $\mathbf{J} = \partial\mathbf{F}^{\mathrm{c}}/\partial\mathbf{u}$. At the same time, using (A21), it is easy to prove that the scheme is conditionally TVD, having as sufficient condition:

$$\mathrm{CFL} \leqslant v \leqslant 1 \tag{23}$$

more restrictive, as expected, than the stability condition (21).

The error introduced by the artificial viscosity v is of second order in space, hence, the scheme is first order accurate in space and time with a truncation error:

$$E = -\frac{\partial^2\mathbf{u}}{\partial t^2}\frac{\Delta t^2}{2} + v\frac{\partial^2\mathbf{u}}{\partial x^2}\frac{\Delta x^2}{2} + O(\Delta t^3, \Delta t\Delta x^2, \Delta x^4) \tag{24}$$

When applied to a homogeneous steady problem, the steady solution is obtained from

$$\mathbf{F}_{i+1}^n = \mathbf{F}_{i-1}^n + \frac{v}{\Delta t}(\delta\mathbf{u}_{i+(1/2)}^n - \delta\mathbf{u}_{i-(1/2)}^n)$$

so that v produces also a second order error in the steady solution. With source terms discretized pointwise, the steady solution comes from

$$\mathbf{F}_{i+1}^n = \mathbf{F}_{i-1}^n + 2\mathbf{S}_i^n\Delta x + \frac{v}{\Delta t}(\delta\mathbf{u}_{i+(1/2)}^n - \delta\mathbf{u}_{i-(1/2)}^n)$$

and with non-centred source terms:

$$\mathbf{F}_{i+1}^n = \mathbf{F}_{i-1}^n + (\mathbf{S}_{i+(1/2)}^n + \mathbf{S}_{i-(1/2)}^n)\Delta x + \frac{v}{\Delta t}(\delta\mathbf{u}_{i+(1/2)}^n - \delta\mathbf{u}_{i-(1/2)}^n)$$

being both first order rules of integration.

Finally, Lax–Friedrichs is also a conservative scheme since, following the definitions given in Appendix A, it admits a nodal flux and a wave decomposition in the form

$$\mathbf{F}_i^{\mathrm{T}} = \mathbf{F}_i^n$$

$$\delta\mathbf{F}_{i+(1/2)}^{\mathrm{L}} = \frac{1}{2}\delta\mathbf{F}_{i+(1/2)}^n + \frac{v\Delta x}{2\Delta t}\delta\mathbf{u}_{i+(1/2)}^n$$

$$\delta\mathbf{F}_{i+(1/2)}^{\mathrm{R}} = \frac{1}{2}\delta\mathbf{F}_{i+(1/2)}^n - \frac{v\Delta x}{2\Delta t}\delta\mathbf{u}_{i+(1/2)}^n$$

or a numerical flux as

$$\mathbf{F}^*_{i+(1/2)} = \frac{1}{2}\left(\mathbf{F}^n_i + \mathbf{F}^n_{i+1} - v\,\frac{\Delta x}{\Delta t}\,\delta\mathbf{u}^n_{i+(1/2)}\right)$$

4.2. Optimized Lax–Friedrichs scheme

Both the stability analysis (21) and the TVD analysis (23) applied to Lax–Friedrichs scheme (17) lead to the conclusion that we are interested in values of v smaller than but close to unity. On the other hand, the truncation error (24) tells us that the spatial error in this method is directly proportional to v. A compromise becomes necessary to enforce global conditions as close as possible.

In this work, a modification of Lax–Friedrichs scheme is proposed so that the parameter v is locally defined. To preserve the conservation property, the following wave decomposition is proposed:

$$\begin{aligned}
\mathbf{F}^{\mathrm{T}}_i &= \mathbf{F}^n_i \\
\delta\mathbf{F}^{\mathrm{L}}_{i+(1/2)} &= \frac{1}{2}\,\delta\mathbf{F}^n_{i+(1/2)} + \frac{\Delta x}{2\Delta t}\,v^n_{i+(1/2)}\delta\mathbf{u}^n_{i+(1/2)} \\
\delta\mathbf{F}^{\mathrm{R}}_{i+(1/2)} &= \frac{1}{2}\,\delta\mathbf{F}^n_{i+(1/2)} - \frac{\Delta x}{2\Delta t}\,v^n_{i+(1/2)}\delta\mathbf{u}^n_{i+(1/2)}
\end{aligned}$$

or the following numerical flux:

$$\mathbf{F}^*_{i+(1/2)} = \frac{1}{2}\left(\mathbf{F}^n_i + \mathbf{F}^n_{i+1} - \frac{\Delta x}{2\Delta t}\,v^n_{i+(1/2)}\delta\mathbf{u}^n_{i+(1/2)}\right)$$

In this new version of the scheme, v, instead of being a constant parameter, is a variable parameter defined at the interior of every cell $i + \frac{1}{2}$. The resulting scheme with pointwise source term is,

$$\begin{aligned}
(1 - \theta\Delta t\mathbf{K}^n_i)\Delta\mathbf{u}^n_i = {}& \frac{1}{2}\,(v^n_{i+(1/2)}\delta\mathbf{u}^n_{i+(1/2)} - v^n_{i-(1/2)}\delta\mathbf{u}^n_{i-(1/2)}) + \Delta t\mathbf{S}^n_i \\
& - \frac{\Delta t}{2}\left[\left(\frac{\delta\mathbf{F}}{\delta x}\right)^n_{i-(1/2)} + \left(\frac{\delta\mathbf{F}}{\delta x}\right)^n_{i+(1/2)}\right]
\end{aligned} \tag{25}$$

and with non-centred source term

$$(1 - \theta\Delta t\mathbf{K}^n_i)\Delta\mathbf{u}^n_i = \frac{1}{2}\,(v^n_{i+(1/2)}\delta\mathbf{u}^n_{i+(1/2)} - v^n_{i-(1/2)}\delta\mathbf{u}^n_{i-(1/2)}) + \frac{\Delta t}{2}\,(\mathbf{G}^n_{i-(1/2)} + \mathbf{G}^n_{i+(1/2)}) \tag{26}$$

where $\mathbf{G}$ can be defined in any of the equivalent forms, conservative (A7), quasi-conservative (A8) or non-conservative (A9). By locally taking the smallest possible value of v compatible with stability, the error is minimized:

$$v^n_{i+(1/2)} = \left[\max(|\lambda_k|^n_{i+1}, |\lambda_k|^n_i)\,\frac{\Delta t}{\Delta x}\right]^2 \tag{27}$$

where λ_k are the eigenvalues of the conservative flux Jacobian. In order to prevent oscillations, the minimum value of v compatible with (23) gives

$$v^n_{i+(1/2)} = \max(|\lambda_k|^n_{i+1}, |\lambda_k|^n_i)\,\frac{\Delta t}{\Delta x} \tag{28}$$

and, in any case, for stability and TVDness, the scheme must follow

$$\mathrm{CFL} \leqslant 1$$

Note that, in the scalar case, condition (27) makes this scheme equivalent to Lax–Wendroff scheme and condition (28) makes it equivalent to the first order upwind scheme. We shall come back to this point and show examples in later sections.

The truncation error of the modified scheme is

$$\mathbf{E} = \left[\left(\theta - \frac{1}{2}\right)\frac{\partial \mathbf{S}}{\partial t} - \frac{1}{2}\frac{\partial^2 \mathbf{F}}{\partial t \partial x}\right]^n_i \Delta t^2 + \frac{\partial}{\partial x}\left(\frac{v}{2}\frac{\partial \mathbf{u}}{\partial x}\right)\Delta x^2 + O(\Delta t^3, \Delta t \Delta x^2, \Delta x^4)$$

When applying this scheme to the shallow water equations, apart from the necessity of a careful choice of the artificial viscosity as described above, a new difficulty appears. An error in conservation may be introduced by the viscosity in the convergence to the steady state. To explain this, first let us consider the truncation error of the scheme in steady state:

$$\mathbf{E} \approx \frac{\partial}{\partial x}\left(\frac{v}{2}\frac{\partial \mathbf{u}}{\partial x}\right)\Delta x^2 = \frac{1}{2}\frac{\partial}{\partial x}\left[v\begin{pmatrix}\frac{\partial A}{\partial x}\\ \frac{\partial Q}{\partial x}\end{pmatrix}\right]$$

and the fact that, the steady state system of equations is

$$\frac{\partial Q}{\partial x} = 0$$

$$S - \frac{\partial F}{\partial x} = 0$$

with F and S the flux and source term, respectively, in the momentum equation.

$$F = \frac{Q^2}{A} + gI_1$$

$$S = E + g\left(I_2 - A\,\frac{\mathrm{d}z_\mathrm{b}}{\mathrm{d}x}\right)$$

This means that the numerical error disappears from the momentum equation as the solution converges to the steady state but not from the mass equation when A changes with x. The idea proposed in this work is to adapt the scheme so that, keeping the previous properties, it makes the conservation error tend to zero during convergence to the steady state.

Differencing the physical flux F we get

$$\frac{\partial F}{\partial x} = (c^2 - u^2)\,\frac{\partial A}{\partial x} + 2u\,\frac{\partial Q}{\partial x}$$

hence the following expression is obtained:

$$\frac{\partial A}{\partial x} = \frac{1}{c^2 - u^2}\left(\frac{\partial F}{\partial x} - 2u\,\frac{\partial Q}{\partial x}\right)$$

which does not vanish in steady state. The following can be used instead:

$$\frac{\partial A}{\partial x} + \frac{S}{u^2 - c^2} = \frac{1}{u^2 - c^2}\left(S - \frac{\partial F}{\partial x} + 2u\,\frac{\partial Q}{\partial x}\right)$$

which tends to zero as the solution converges to steady state. Unfortunately, this term is singular in transcritical cases being the error introduced inversely proportional to $u^2 - c^2$, which can become very important near critical state. Two possibilities are envisaged for the discretization of the mass equation, leading to the following optimized Lax–Friedrichs scheme (OLF).

$$(1 - \theta\Delta t\mathbf{K}_i^n)\Delta\mathbf{u}_i^n = \frac{1}{2}\,(v_{i+(1/2)}^n\delta\mathbf{v}_{i+(1/2)}^n - v_{i-(1/2)}^n\delta\mathbf{v}_{i-(1/2)}^n) + \frac{\Delta t}{2}\,(\mathbf{G}_{i-(1/2)}^n + \mathbf{G}_{i+(1/2)}^n) \tag{29}$$

where

$$\delta\mathbf{v}_{i+(1/2)} = \begin{pmatrix} \text{modmin}\,(\delta A, \frac{1}{u^2-c^2}\,(S\delta x - \delta F + 2u\delta Q)) \\ \delta Q \end{pmatrix}_{i+(1/2)}$$

and the following function is defined:

$$\text{modmin}(f, g) = \begin{cases} 0 & \text{if } fg \leqslant 0 \\ f & \text{if } |f| < |g| \quad \text{and} \quad fg > 0 \\ g & \text{if } |f| \geqslant |g| \quad \text{and} \quad fg > 0 \end{cases} \tag{30}$$

The performance of the modified Lax–Friedrichs scheme when supplied with the above artificial viscosity is shown in the section corresponding to numerical results.

For the sake of completness, in order to provide a reference and all the information necessary to compare the properties and performance of Lax–Friedrichs scheme, the first order upwind and the second order Lax–Wendroff schemes are summarized in the following subsections.

4.3. First order upwind scheme

Upwind schemes are based in a non-centred approximation of the spatial derivatives according to the sense of propagation of information in the equation. Therefore they are known to offer a discrete solution closer to the physical process than a centred scheme. In order to consider the influence of all the possible propagation velocities, preserving at the same time the conservative character of the scheme, the characteristic form of the conservative schemes (A14) can be used. In explicit form with semi-implicit source term discretization becomes

$$(1 - \theta\Delta t\mathbf{K}_i^n)\Delta\mathbf{u}_i^n = \Delta t[(\mathbf{P}\mathbf{\Omega}^{\mathrm{L}}\mathbf{P}^{-1}\mathbf{G})_{i-(1/2)}^n + (\mathbf{P}\mathbf{\Omega}^{\mathrm{R}}\mathbf{P}^{-1}\mathbf{G})_{i+(1/2)}^n]$$

where $\mathbf{G}$ can be expressed in any of the equivalent forms previously defined. The upwind character of the scheme is given by the following definition of the matrices:

$$\begin{aligned} \boldsymbol{\Omega}^{\mathrm{L}} &= \boldsymbol{\Omega}^{+} = \frac{1}{2}\,[\mathbf{I} + \operatorname{sign}(\boldsymbol{\Lambda})] \\ \boldsymbol{\Omega}^{\mathrm{R}} &= \boldsymbol{\Omega}^{-} = \frac{1}{2}\,[\mathbf{I} - \operatorname{sign}(\boldsymbol{\Lambda})] \\ \mathbf{G}^{\pm} &= \mathbf{P}\boldsymbol{\Omega}^{\pm}\mathbf{P}^{-1}\mathbf{G} \end{aligned} \tag{31}$$

From which the first order explicit upwind scheme with semi-implicit and upwind source term discretization follows:

$$(1 - \theta\Delta t\mathbf{K}_i^n)\Delta\mathbf{u}_i^n = \Delta t[(\mathbf{G}^{+})_{i-(1/2)}^n + (\mathbf{G}^{-})_{i+(1/2)}^n] \tag{32}$$

Pointwise source term discretization leads to

$$(1 - \theta\Delta t\mathbf{K}_i^n)\Delta\mathbf{u}_i^n = \Delta t\left[\mathbf{S}_i^n - \left(\frac{\delta\mathbf{F}^{+}}{\delta x}\right)_{i-(1/2)}^n - \left(\frac{\delta\mathbf{F}^{-}}{\delta x}\right)_{i+(1/2)}^n\right] \tag{33}$$

where

$$\delta\mathbf{F}^{\pm} = \mathbf{P}\boldsymbol{\Omega}^{\pm}\mathbf{P}^{-1}\delta\mathbf{F} \tag{34}$$

Being a three point scheme of the type (A15) with coefficients, in the linear case:

$$A = B = 0, \quad C = -\frac{a\Delta t}{2\Delta x}, \quad D = \left|\frac{a\Delta t}{2\Delta x}\right|$$

it is easy to verify that using (A18) and (A21), the scheme is stable and TVD when

$$\mathrm{CFL} \leqslant 1$$

According to previous definitions, the conservative character of the scheme is proved by the existence of a nodal flux and wave decomposition

$$\mathbf{F}_i^{\mathrm{T}} = \mathbf{F}_i^n$$

$$\delta\mathbf{F}_{i+(1/2)}^{\mathrm{L}} = (\delta\mathbf{F}^{+})_{i+(1/2)}^n \qquad \delta\mathbf{F}_{i+(1/2)}^{\mathrm{R}} = (\delta\mathbf{F}^{-})_{i+(1/2)}^n$$

and a numerical flux:

$$\mathbf{F}_{i+(1/2)}^{*} = \mathbf{F}_i^{\mathrm{T}} + \delta\mathbf{F}_{i+(1/2)}^{\mathrm{R}} = \frac{1}{2}\,[\mathbf{F}_i^n + \mathbf{F}_{i+1}^n - |\mathbf{J}|_{i+(1/2)}^n\,\delta\mathbf{u}_{i+(1/2)}^n]$$

with $|\mathbf{J}| = \mathbf{P}|\boldsymbol{\Lambda}|\mathbf{P}^{-1}$.

The resulting scheme is unable to deal with subcritical-supercritical transitions, requiring the addition of extra viscosity at those points. The correction to that problem preserving conservation requires the following modification of the wave decomposition

$$\begin{gathered} \mathbf{F}_i^{\mathrm{T}} = \mathbf{F}_i^n, \quad \delta\mathbf{F}_{i+(1/2)}^{\mathrm{L}} = (\delta\mathbf{F}^{+})_{i+(1/2)}^n + v_{i+(1/2)}^n\,\delta\mathbf{u}_{i+(1/2)}^n, \\ \delta\mathbf{F}_{i+(1/2)}^{\mathrm{R}} = (\delta\mathbf{F}^{-})_{i+(1/2)}^n - v_{i+(1/2)}^n\delta\mathbf{u}_{i+(1/2)}^n \end{gathered} \tag{35}$$

and the following of the numerical flux:

$$\mathbf{F}^*_{i+(1/2)} = \mathbf{F}^{\mathrm{T}}_i + \delta\mathbf{F}^{\mathrm{R}}_{i+(1/2)} = \frac{1}{2}\,[\mathbf{F}^n_i + \mathbf{F}^n_{i+1} - |\mathbf{J}|^n_{i+(1/2)}\delta\mathbf{u}^n_{i+(1/2)}] - v^n_{i+(1/2)}\,\delta\mathbf{u}^n_{i+(1/2)} \tag{36}$$

The particular way to introduce this artificial viscosity or entropy correction in this work is

$$v^n_{i+(1/2)} = \begin{cases} \frac{1}{4}\delta(\lambda^k)_{i+(1/2)} & \text{if } (\lambda^k)^n_i < 0 \text{ and } (\lambda^k)^n_{i+1} > 0 \\ 0 & \text{otherwise} \end{cases} \tag{37}$$

4.4. *Lax–Wendroff scheme*

The original idea from Lax and Wendroff was proposed for equations without source terms and is based on a Taylor series of the conservation law to second order in time:

$$\begin{aligned}
\mathbf{u}^{n+1}_i &= \mathbf{u}^n_i - \left(\frac{\partial \mathbf{F}}{\partial x}\right)^n_i \Delta t - \frac{\partial}{\partial x}\left(\frac{\partial \mathbf{F}}{\partial t}\right)^n_i \frac{\Delta t^2}{2} + O(\Delta t^3) \\
&= \mathbf{u}^n_i - \left(\frac{\partial \mathbf{F}}{\partial x}\right)^n_i \Delta t - \frac{\partial}{\partial x}\left(\mathbf{J}\,\frac{\partial \mathbf{u}}{\partial t}\right)^n_i \frac{\Delta t^2}{2} + O(\Delta t^3) \\
&= \mathbf{u}^n_i - \left(\frac{\partial \mathbf{F}}{\partial x}\right)^n_i \Delta t + \frac{\partial}{\partial x}\left(\mathbf{J}\,\frac{\partial \mathbf{F}}{\partial x}\right)^n_i \frac{\Delta t^2}{2} + O(\Delta t^3)
\end{aligned}$$

together with a central discretization of the spatial derivatives

$$\Delta\mathbf{u}^n_i = -\frac{\Delta t}{2}\left[\left(1 + \frac{\Delta t}{\Delta x}\,\mathbf{J}^n_{i-(1/2)}\right)\left(\frac{\delta\mathbf{F}}{\delta x}\right)^n_{i-(1/2)} + \left(1 - \frac{\Delta t}{\Delta x}\,\mathbf{J}^n_{i+(1/2)}\right)\left(\frac{\delta\mathbf{F}}{\delta x}\right)^n_{i+(1/2)}\right] \tag{38}$$

and produces a method of second order in space and time [18].

It is a 3 point scheme with coefficients, in the linear case

$$A = B = 0, \quad C = -\frac{a\Delta t}{2\Delta x}, \quad D = \frac{a^2\Delta t^2}{2\Delta x^2}$$

hence, applying (A18), the stability condition of the scheme is

$$\mathrm{CFL} \leqslant 1$$

Using on the other hand (A21), it is clear that the scheme cannot be ensured to be TVD.

Lax–Wendroff scheme is conservative with a nodal flux and wave decomposition:

$$\begin{aligned}
\mathbf{F}^{\mathrm{T}}_i &= \mathbf{F}^n_i \\
\delta\mathbf{F}^{\mathrm{L}}_{i+(1/2)} &= \frac{1}{2}\left(1 + \frac{\Delta t}{\Delta x}\,\mathbf{J}\right)^n_{i+(1/2)} \delta\mathbf{F}^n_{i+(1/2)} \\
\delta\mathbf{F}^{\mathrm{R}}_{i+(1/2)} &= \frac{1}{2}\left(1 - \frac{\Delta t}{\Delta x}\,\mathbf{J}\right)^n_{i+(1/2)} \delta\mathbf{F}^n_{i+(1/2)}
\end{aligned}$$

and a numerical flux

$$\mathbf{F}^*_{i+(1/2)} = \frac{1}{2}\left[\mathbf{F}^n_i + \mathbf{F}^n_{i+1} - \frac{\Delta t}{\Delta x}(\mathbf{J}\delta\mathbf{F})^n_{i+(1/2)}\right]$$

The scheme is not able to deal properly with subcritical to supercritical transitions as it happens to the first order upwind, requiring also the addition of artificial viscosity. The use of a viscosity as defined in (37) and further modification of the wave decomposition like in (35) or the numerical flux like in (36) cures the problem.

To adapt the Lax–Wendroff scheme to conservation laws with source terms, a new Taylor series must be performed using the complete equation [19]:

$$\begin{aligned}\mathbf{u}^{n+1}_i &= \mathbf{u}^n_i + \left(\mathbf{S} - \frac{\partial\mathbf{F}}{\partial x}\right)^n_i \Delta t + \frac{\partial}{\partial t}\left(\mathbf{S} - \frac{\partial\mathbf{F}}{\partial x}\right)^n_i \frac{\Delta t^2}{2} + O(\Delta t^3)\\ &= \mathbf{u}^n_i + \left(\mathbf{S} - \frac{\partial\mathbf{F}}{\partial x}\right)^n_i \Delta t + \left\{\mathbf{K}\frac{\partial\mathbf{u}}{\partial t} - \frac{\partial}{\partial x}\left[\mathbf{J}\left(\mathbf{S} - \frac{\partial\mathbf{F}}{\partial x}\right)\right]\right\}^n_i \frac{\Delta t^2}{2} + O(\Delta t^3) \qquad (39)\end{aligned}$$

Then, a central discretization of the spatial derivatives leads to the extended Lax–Wendroff scheme with pointwise and semi-implicit source terms discretization:

$$\begin{aligned}\left(1 - \frac{1}{2}\Delta t\mathbf{K}^n_i\right)\Delta\mathbf{u}^n_i = \mathbf{S}^n_i\Delta t - \frac{\Delta t}{2}\left[\left(\frac{\delta\mathbf{F}}{\delta x}\right)^n_{i-(1/2)} + \left(\frac{\delta\mathbf{F}}{\delta x}\right)^n_{i+(1/2)}\right]\\ - \frac{\Delta t^2}{2\Delta x}[(\mathbf{J}\mathbf{G})^n_{i+(1/2)} - (\mathbf{J}\mathbf{G})^n_{i-(1/2)}] \qquad (40)\end{aligned}$$

or with centred and semi-implicit source terms discretization:

$$\left(1 - \frac{1}{2}\Delta t\mathbf{K}^n_i\right)\Delta\mathbf{u}^n_i = \frac{\Delta t}{2}\left[\left(1 + \frac{\Delta t}{\Delta x}\mathbf{J}^n_{i-(1/2)}\right)\mathbf{G}^n_{i-(1/2)} + \left(1 - \frac{\Delta t}{\Delta x}\mathbf{J}^n_{i+(1/2)}\right)\mathbf{G}^n_{i+(1/2)}\right] \qquad (41)$$

Note that the centred discretization of the source terms is different from the upwind discretization and also different from the pointwise discretization. In Lax–Wendroff scheme, only the centered discretization provides the equilibrium between fluxes and sources at the discrete level.

5. NUMERICAL RESULTS AND DISCUSSION

The performance of the different methods when applied to the test cases will be shown next. Figure 4 displays the comparison of the numerical and exact solution to the linear advection of the initial square profile. Figure 4(a) is the result of applying Lax–Friedrichs scheme (LF) with maximum viscosity ($v = 1$) in 100 cells, and the classical 'rough' tendency is noticeable. The roughness disappears when the global viscosity is reduced to $v = 0.9$ as displayed in Figure 4(b), where, despite 400 cells have been used, the numerical diffusion associated to the method is obvious. Figures 4(c) and 4(d) correspond to the TVD and stable versions of the Optimized Lax–Friedrichs scheme (OLF) respectively, and Figures 4(e) and 4(f) are the

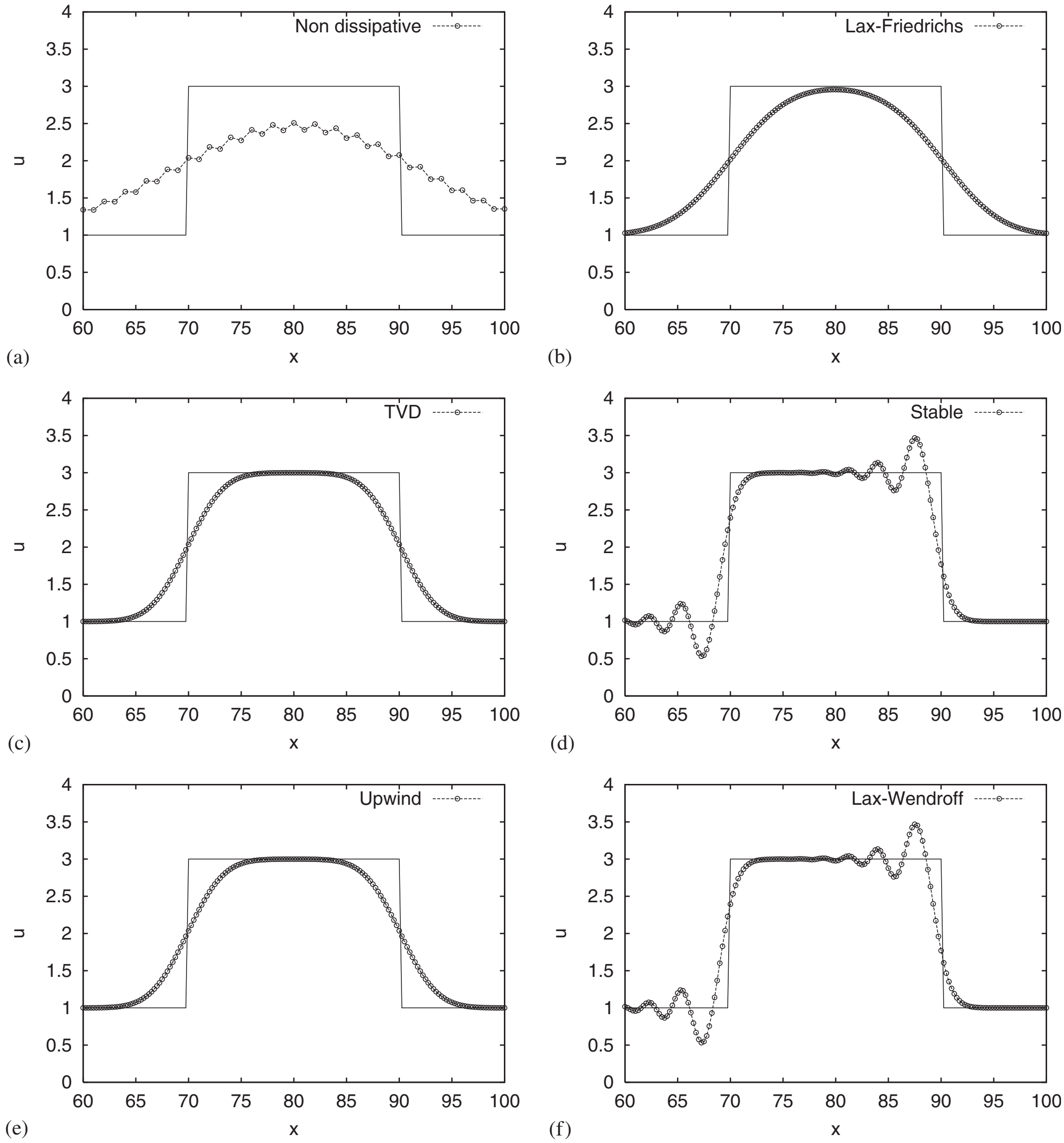


Figure 4. Linear advection of a square using LF with $\nu = 1$ (a) and with $\nu = 0.9$ (b), TVD OLF (c), stable OLF (d), upwind (e) and LW (f) with, $\mathrm{CFL} = 0.5$.

results from the first order upwind and Lax–Wendroff (LW) scheme respectively. It is clear that in this case the TVD OLF is equivalent to the first order upwind and the stable OLF to the LW scheme, these showing oscillations near the strong gradient.

The same can be observed for the linear advection of an initial gaussian profile. This is displayed in Figure 5. In this case, the stable OLF and LW methods do not show oscillations

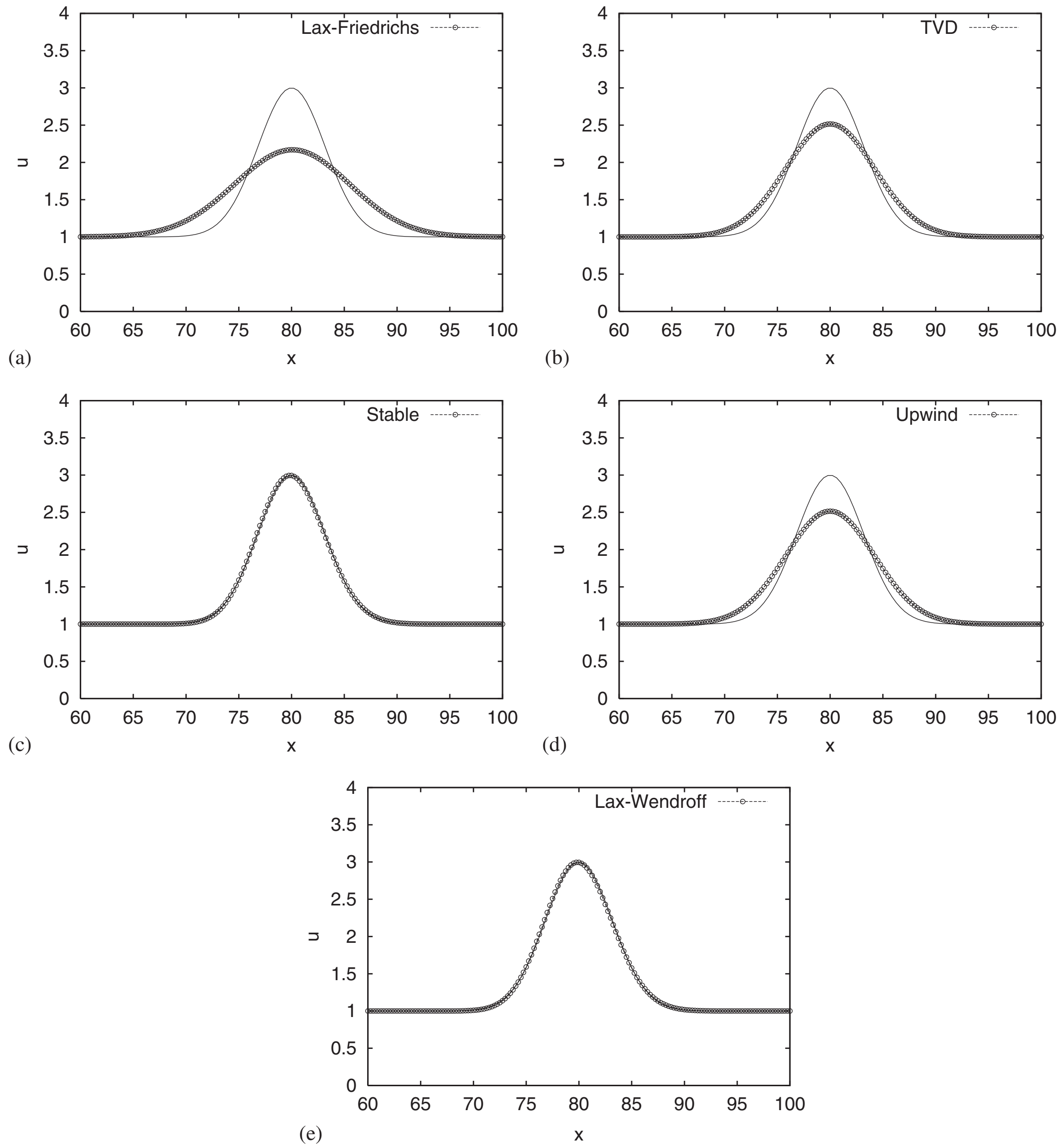


Figure 5. Linear advection of a gaussian using LF (a), TVD OLF (b), stable OLF (c), first order upwind (d) and LW (e) with, CFL $=0.5$.

due to the smoothness of the profile and they are more accurate than TVD OLF and first order upwind.

Figure 6 corresponds to the exact and numerical results for the Burgers test case. Only the OLF, upwind and LW schemes are compared here. It is worth noting that, due to the initial values in this test case, a sub-super transcritical point is present in the solution. The

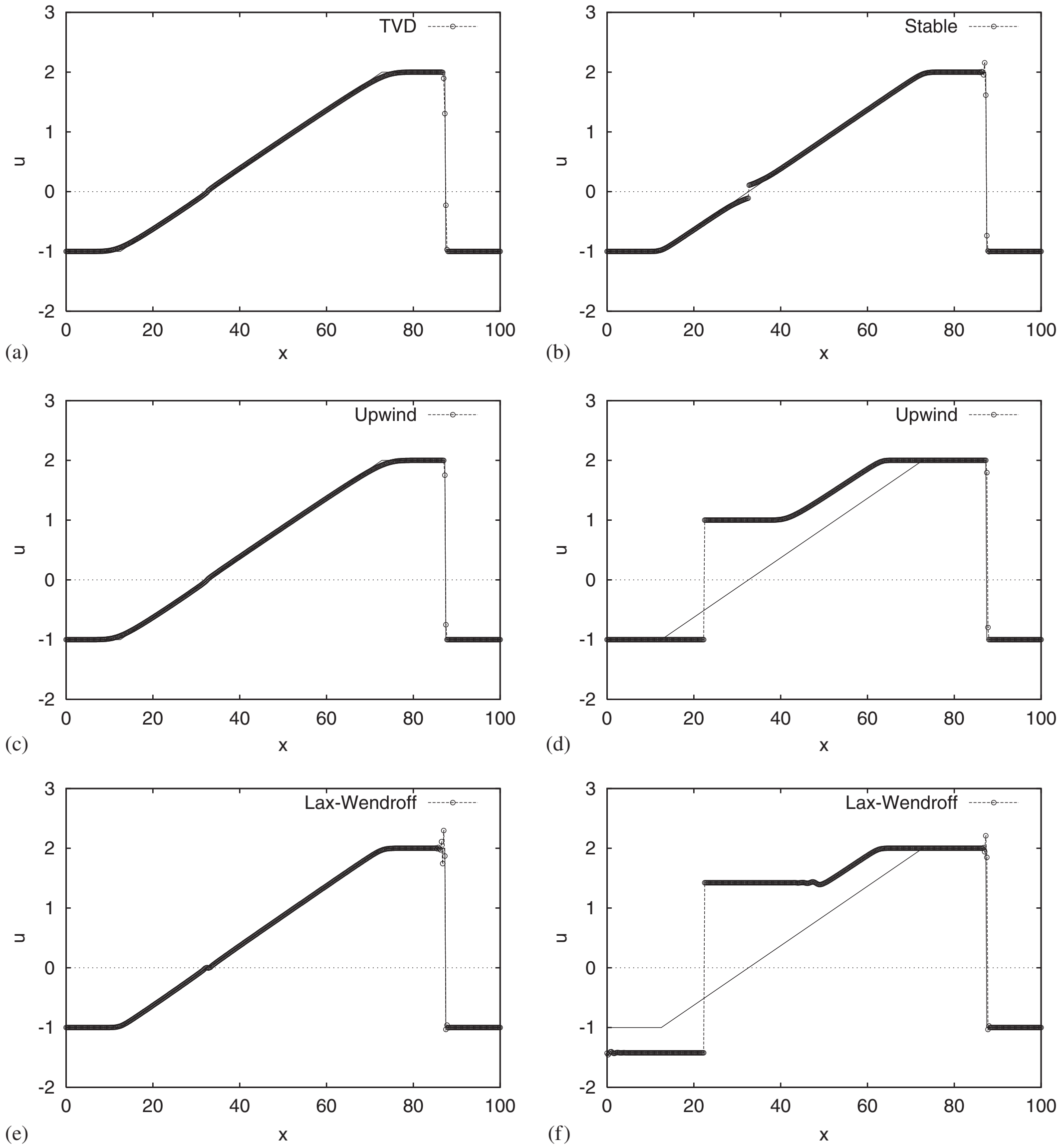


Figure 6. Inviscid Burgers solution using TVD OLF (a), stable OLF (b), first order upwind with (c) and without (d) entropy correction, LW with (e) and without (f) entropy correction.

results show the inaccuracy of the first order upwind and LW when dealing with this transition in Figures 6(d) and 6(f) and how the addition of artificial viscosity cures the problem in (c) and (e) as announced. It is also interesting to note that in this non-linear problem the TVD OLF and stable OLF results are similar to the entropy corrected upwind and LW schemes.

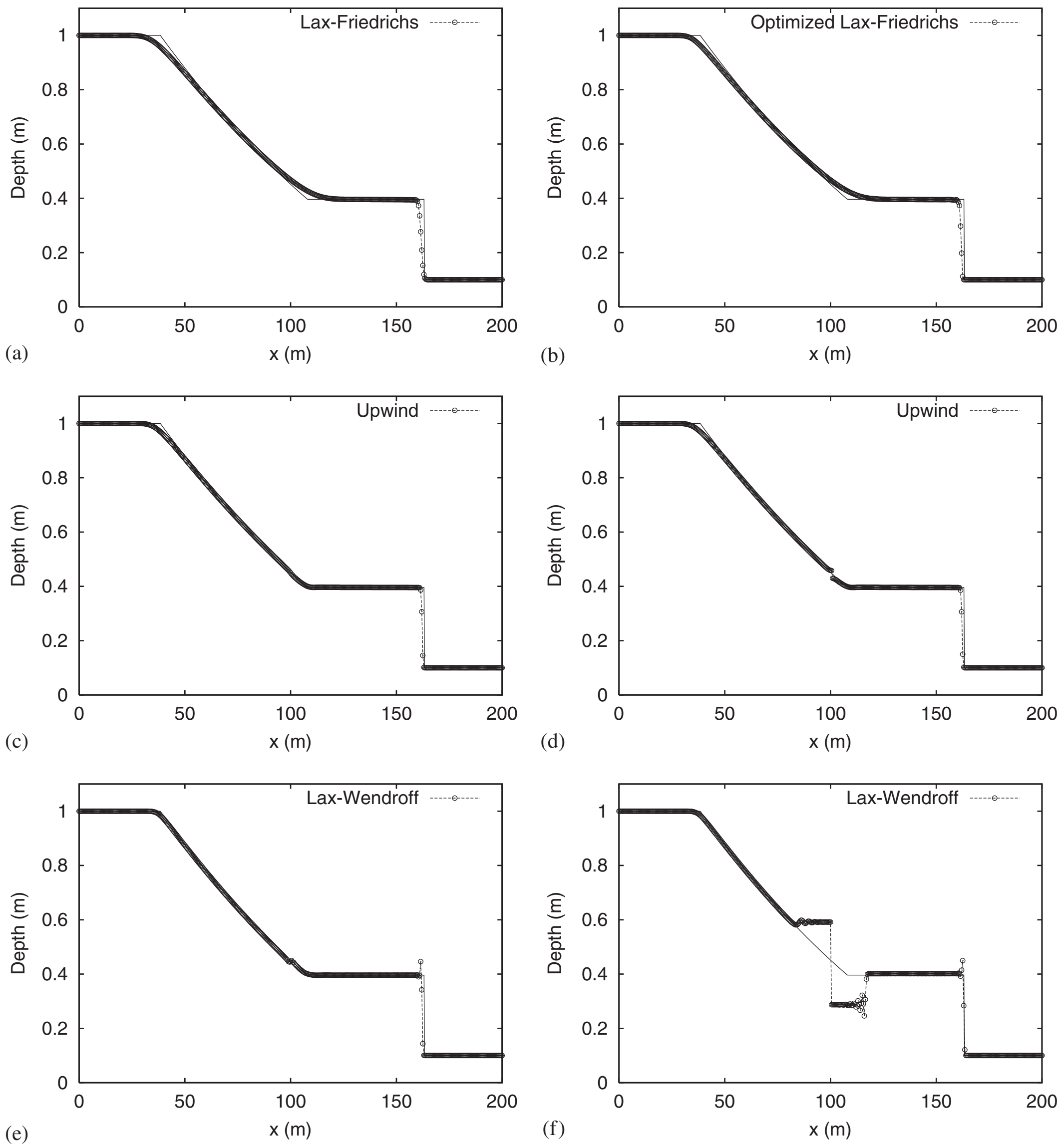


Figure 7. Dam break solution using LF (a), OLF (b), first order upwind with (c) and without (d) entropy correction, LW with (e) and without (f) entropy correction.

Figure 7 displays the exact and numerical solutions to the dam break test case. The basic LF scheme produces a reasonable result in this case. From now on, only the TVD OLF will be used for monotonicity reasons. It is displayed in Figure 7(b). The numerical problems at the critical transition are also present in Figures 7(d) and 7(f) and solved by controlled addition of artificial viscosity Figures 7(c) and 7(e).

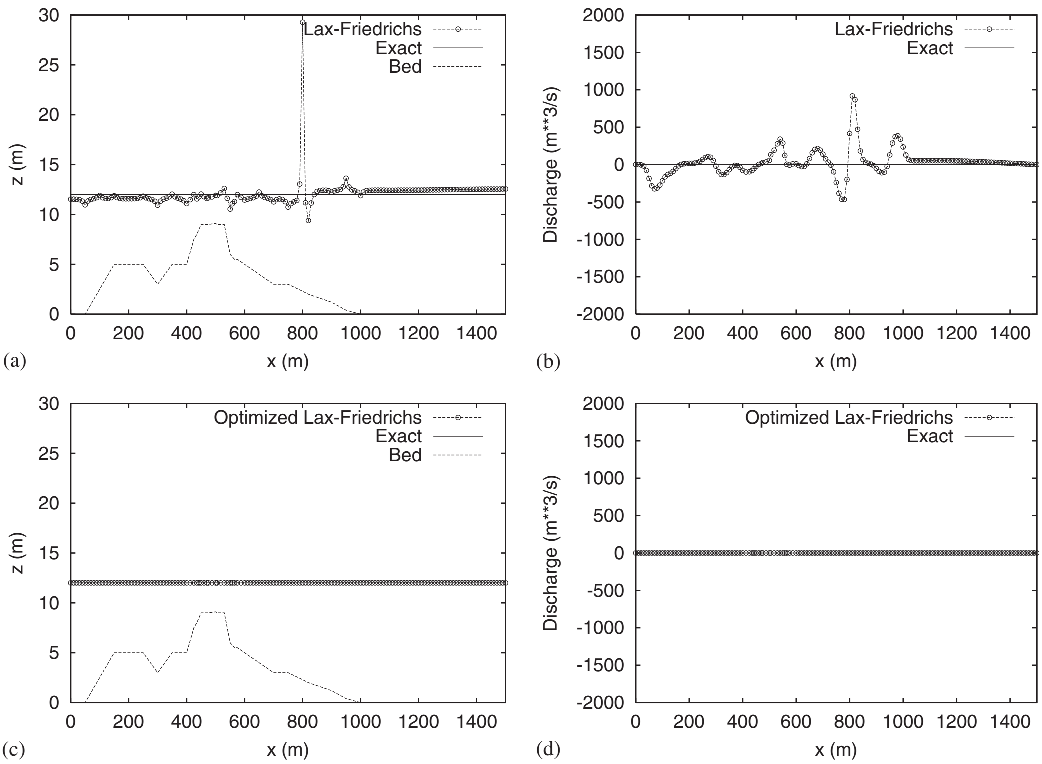


Figure 8. Water surface and discharge profile in Goutal and Maurel's test case using LF (a), (b) and OLF (c), (d).

Figures 8 and 9 show the exact and numerical solution to the Goutal–Maurel test case by means of the water level and discharge profiles. The exact solution is trivial and corresponds to zero discharge and horizontal water level. This is perfectly reproduced by all the schemes except the basic LF and the LW with pointwise source term (40). In the case of the first order upwind scheme, the source terms have been decomposed like in (32) leading to the correct solution as expected from previous works on upwind schemes. This is a good example to demonstrate, on the one hand, the importance of an adequate source term discretization both in upwind and central schemes and, on the other hand, the bad consequences that a careless artificial viscosity can generate in presence of relevant source terms on the solutions of an apparently simple numerical scheme.

Figures 2 and 10 are the representations of the exact and numerical water depth and discharge profiles at steady state for the MacDonald-1 test case. It is a subcritical and smooth solution and, therefore, there are not noticeable differences among the OLF, upwind and LW results. However the LF scheme is again inaccurate at steady state. The results corresponding to MacDonald-2 test case are presented in Figures 11 and 12. Again, the improvement introduced in the results by the optimization to LF scheme is remarkable. Being a solution with several critical transitions, only the entropy corrected upwind and LW schemes have been used. Typical discharge oscillations at the hydraulic jump location are present in all schemes.

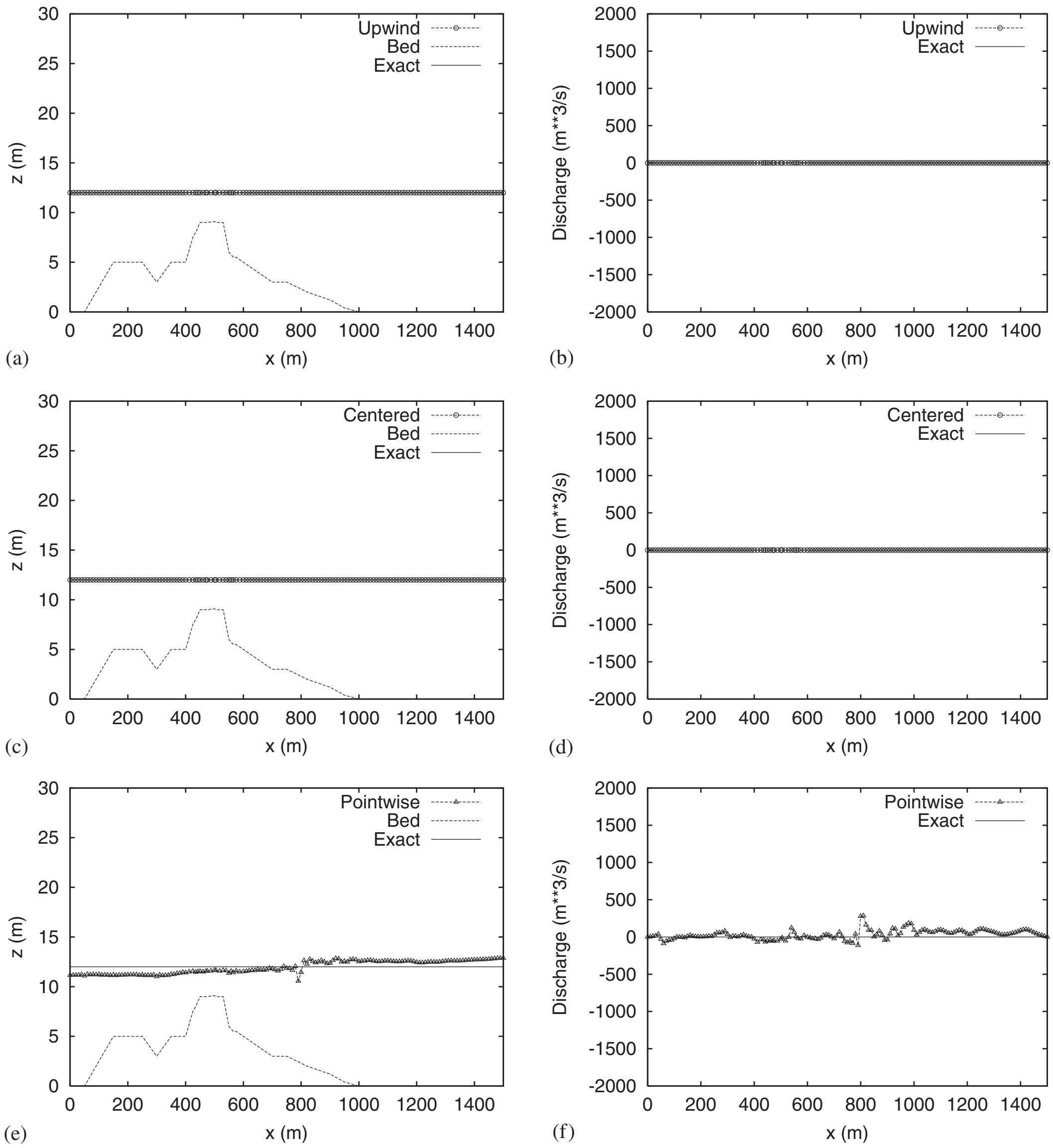


Figure 9. Water surface and discharge profile in Goutal and Maurel's test case using upwind (a), (b), LW with centred source terms (c), (d) and LW with pointwise source terms (e), (f).

6. APPLICATION TO RIVER FLOW

The numerical schemes previously discussed are now applied to a case of practical interest. The river reach used for the simulation belongs to the lower part of the Ebro river (Zaragoza, Spain) and therefore has very mild average slope and low water velocities. However, the river

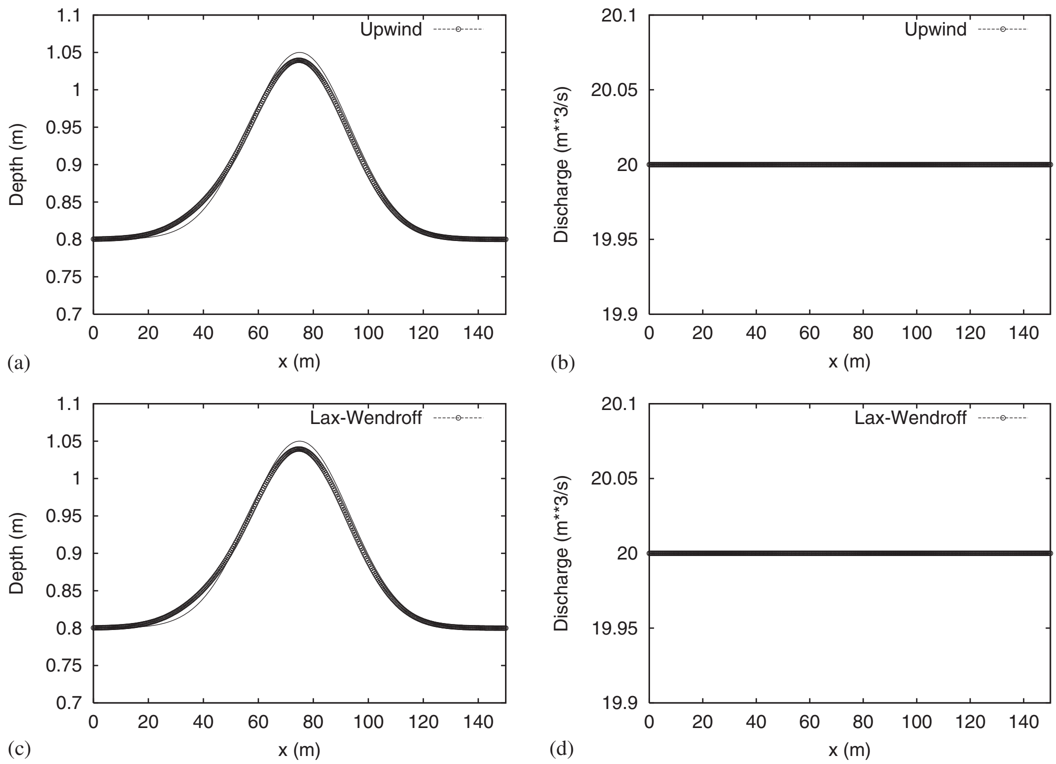


Figure 10. Water depth and discharge at steady state in the MacDonald-1 test case using the first order upwind (a), (b) and LW scheme (c), (d) with $\Delta x = 0.375$ m and CFL $= 0.9$.

cross section is highly variable in shape along the axis of the river and presents an irregular tendency in the bottom level variation leading to adverse and important slopes in some parts. This can be observed when the bottom profile along the river is plotted.

The total length of the simulated reach is around 11.4 km. Geometric data were available at 64 cross sections. As we were mainly interested in testing the performance of the different methods in presence of a gradually varied practical case dominated by the bed topography source terms, the Manning coefficient was assumed uniform and equal to 0.03 in IS.

A first run supplied the initial condition for the flooding simulation. This first run started from dry bed and introduced an upstream constant discharge of $Q = 200\,\mathrm{m^3/s}$ until convergence. Using the converged steady state as base flow, the flood was represented by means of an upstream hydrograph. The shape of this hydrograph was simplified making it triangular and only the peak discharge, $Q = 5300\,\mathrm{m^3/s}$, at $t = 12$ h, corresponded to the estimated maximum discharge for the flood event of return period equal to 500 years. From the numerical point of view, 400 nodal values were used and a CFL $= 0.9$ leading to a CPU time of around 7 min for the 36 h simulated on a Pentium II PC. Figures 13 and 14 display the results of the numerical schemes for the water depth and discharge profiles at base flow, $Q = 200\,\mathrm{m^3/s}$, corresponding to $t = 0$ for the flood wave simulation and $Q = 5300\,\mathrm{m^3/s}$, corresponding to $t = 12$ h in the

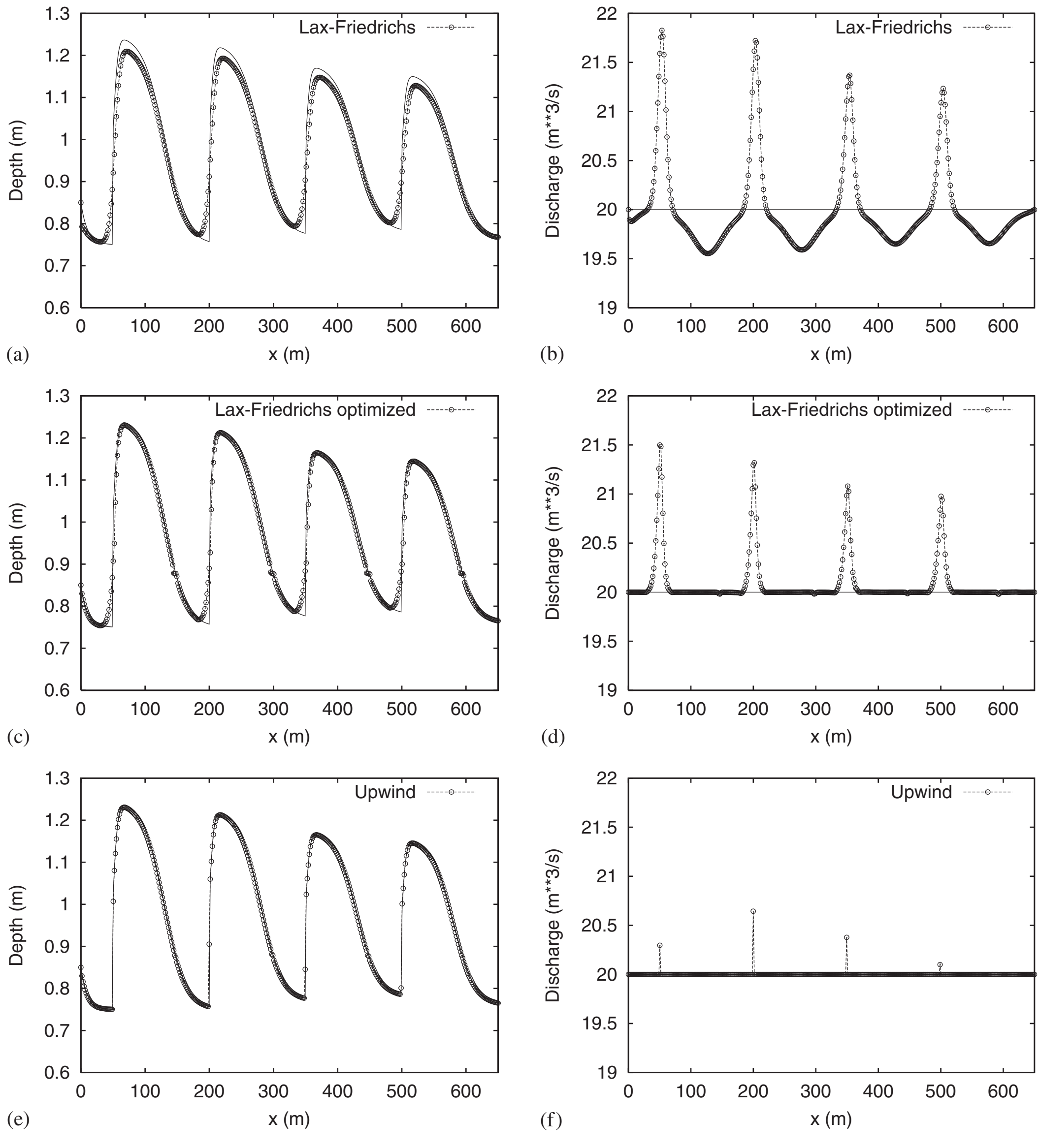


Figure 11. Water depth profile and discharge at steady state for the MacDonald-2 test case using LF (a), (b), OLF (c), (d) and first order upwind (e), (f).

simulation. As it can be observed, only the LF scheme is unable to give smooth and conservative results, being the solution spoiled by the interference of the artificial viscosity and the source terms. On the other hand, all three schemes OLF, upwind and LW give very similar, conservative and well behaved results.

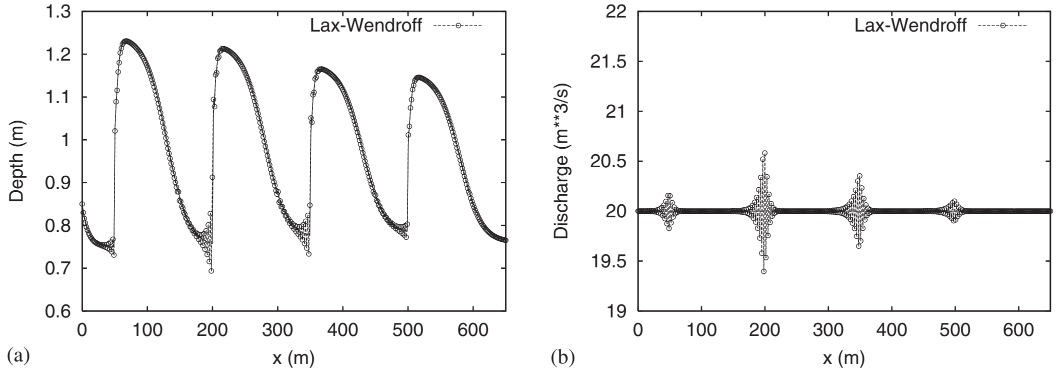


Figure 12. Water depth (a) and discharge (b) profile at steady state in MacDonald-2 test case using LW.

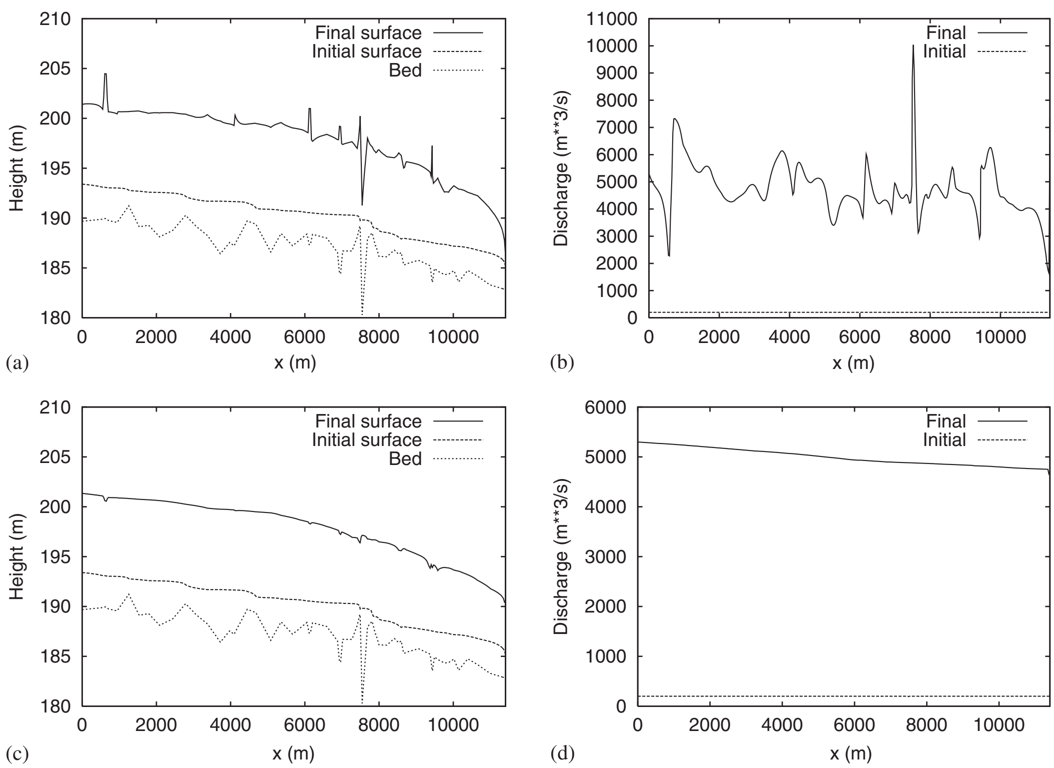


Figure 13. Water surface and discharge profiles at base flow ($t = 0$) and at $t = 12$ h using LF (a), (b) and OLF (c), (d).

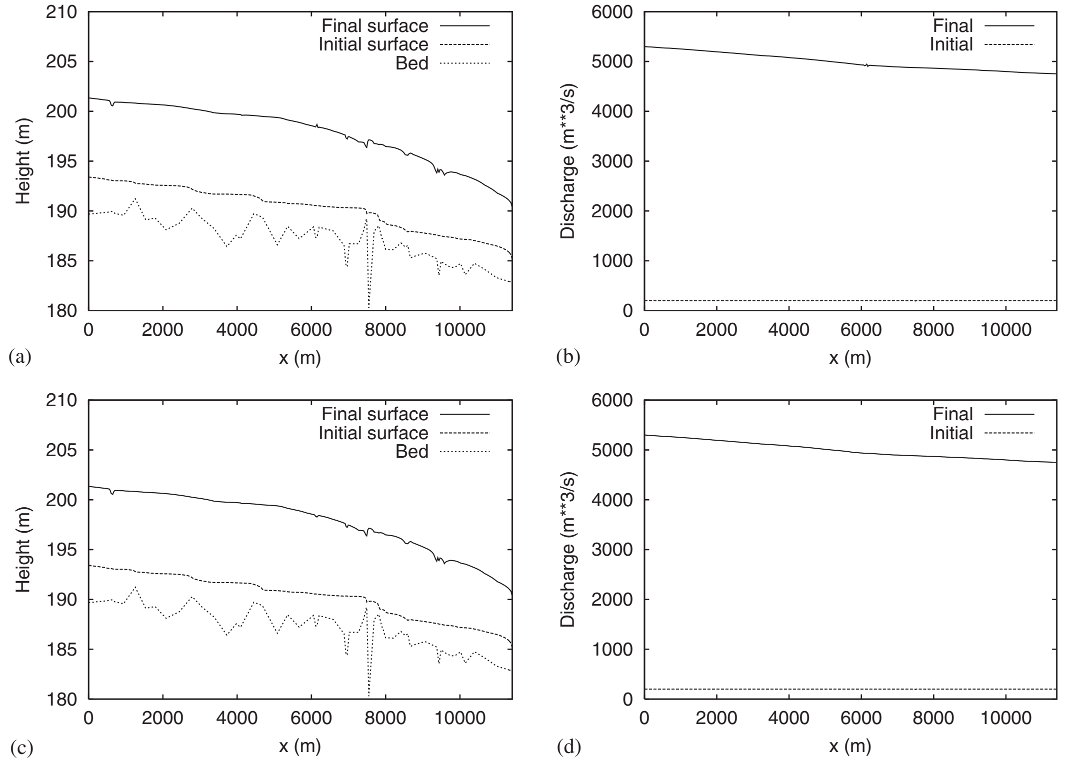


Figure 14. Water surface and discharge profiles at base flow ($t = 0$) and at $t = 12$ h using first order upwind (a), (b) and LW (c), (d).

7. CONCLUSIONS

A rigorous study of the Lax–Friedrichs, first order upwind and Lax–Wendroff scheme has been presented and their properties, according to traditional analysis established.

Lax–Friedrichs scheme is first order in space in time and dissipative due to a global artificial viscosity coefficient. Two possibilities for the definition of a local artificial viscosity have been presented. They lead to performances of the scheme close to those of the first order upwind scheme and the (second order) Lax–Wendroff scheme. The distortion in the solution caused by the interference between the artificial viscosity and the irregularities in the channel geometry has been identified and an optimization of the LF scheme has been proposed.

Lax–Wendroff scheme has been carefully derived for equations with source terms as a generalization of the classical method originally developed for homogeneous conservation laws. The necessity to supply this method with an artificial viscosity in sub-super transition points has been signaled.

Several steady and unsteady one-dimensional examples with exact solution have been used to validate the numerical results provided by the schemes and the suitability of the proposed modifications. According to the solutions obtained, it can be concluded that the optimization proposed for the Lax–Friedrichs scheme strongly improves the performance of this method both in simple and complex cases. At the same time, the modification does not involve extra calculations, keeping the method very simple and intuitive.

The addition of a local artificial viscosity to the first order upwind and to the Lax–Wendroff has been proved necessary in critical transitions and a useful expression for it has been supplied and validated. At the same time, the necessity to balance the discretization of fluxes and sources in Lax–Wendroff scheme, as in upwind schemes, has been pointed out and demonstrated in the test cases.

Finally, the methods have been used for the simulation of a flood wave in a river reach, showing the same trend. In cases of gradually varied river flow, it can be concluded that a first order scheme, if carefully discretized to balance fluxes and sources, gives results similar to a second order scheme. Within first order methods, the choice between an upwind or a central method like the optimized Lax–Friedrichs scheme is open since the latter is considerably simpler and produces almost similar results to the former.

APPENDIX A: SOME PROPERTIES OF THE 1D NUMERICAL SCHEMES

Some interesting and desirable properties will next be defined for the sake of further comparison among the numerical schemes.

A.1. Conservation

Any conservation equation in integral form, when extended to the full domain

$$\int_0^t \mathrm{d}t \int_0^L \mathrm{d}x \left(\frac{\partial \mathbf{u}}{\partial t} + \frac{\mathrm{d}\mathbf{F}}{\mathrm{d}x}\right) = \int_0^t \mathrm{d}t \int_0^L \mathrm{d}x\, \mathbf{S} \Rightarrow \int_0^L [\mathbf{u}(x,t) - \mathbf{u}(x,0)]\, \mathrm{d}x$$
$$= \int_0^t \mathrm{d}t(\mathbf{F}_0 - \mathbf{F}_L) + \int_0^t \mathrm{d}t \int_0^L \mathrm{d}x\, \mathbf{S} \tag{A1}$$

means that the variation in the conserved variable is due to the net balance between the incoming and outgoing fluxes plus the contribution of the source/sink terms. Numerical schemes preserving this important property are called conservative schemes.

The most common definition of a conservative scheme is that accepting the following structure [18]:

$$\Delta \mathbf{u}_i^n = \Delta t \left[\mathbf{S}_i^* - \frac{1}{\partial x}(\mathbf{F}_{i+(1/2)}^* - \mathbf{F}_{i-(1/2)}^*)\right] \tag{A2}$$

where $\mathbf{S}^*$ and $\mathbf{F}^*$ are the numerical source and flux, respectively. They represent a convenient approximation to the true source and flux terms. Using Δ for time increments and δ for space increments, the schemes so defined provide a numerical cancellation of the flux contributions at the internal cell edges, hence generating time variations in the conserved variable only by boundary and source terms influence.

$$\sum_n \sum_i \Delta \mathbf{u}_i^n \delta x \approx \int_0^L [\mathbf{u}(x,t) - \mathbf{u}(x,0)]\,\mathrm{d}x$$

$$\sum_n \sum_i \left[\mathbf{S}_i^* - \frac{1}{\partial x}(\mathbf{F}^*_{i+(1/2)} - \mathbf{F}^*_{i-(1/2)})\right]\Delta t \delta x \approx \int_0^t \mathrm{d}t(\mathbf{F}_0 - \mathbf{F}_L) + \int_0^t \mathrm{d}t \int_0^L \mathrm{d}x\,\mathbf{S}$$

An equivalent form of defining conservative schemes is to define a nodal flux $\mathbf{F}_i^{\mathrm{T}}$ and to decompose in waves the cell flux difference.

$$\begin{aligned}\delta \mathbf{F}^{\mathrm{T}}_{i+(1/2)} &= \delta \mathbf{F}^{\mathrm{R}}_{i+(1/2)} + \delta \mathbf{F}^{\mathrm{L}}_{i+(1/2)} \\ \Delta \mathbf{u}_i^n &= \Delta t \left[\mathbf{S}_i^* - \frac{1}{\delta x}(\delta \mathbf{F}^{\mathrm{R}}_{i+(1/2)} + \delta \mathbf{F}^{\mathrm{L}}_{i-(1/2)})\right]\end{aligned} \tag{A3}$$

There is an equivalence between both formulations:

$$\mathbf{F}^*_{i+(1/2)} = \mathbf{F}_i^{\mathrm{T}} + \delta \mathbf{F}^{\mathrm{R}}_{i+(1/2)} = \mathbf{F}^{\mathrm{T}}_{i+1} - \delta \mathbf{F}^{\mathrm{L}}_{i+(1/2)}$$

The accuracy of the numerical schemes is much improved if the source terms involving spatial derivatives are discretized in the same way [12, 13, 19, 20]. Defining:

$$\mathbf{S}^{\mathrm{T}}_{i+(1/2)} = \mathbf{S}^{\mathrm{R}}_{i+(1/2)} + \mathbf{S}^{\mathrm{L}}_{i+(1/2)}$$

numerical schemes with non-centred source terms can be built:

$$\Delta \mathbf{u}_i^n = \Delta t \left[\left(\mathbf{S} - \frac{\delta \mathbf{F}}{\delta x}\right)^{\mathrm{L}}_{i-(1/2)} + \left(\mathbf{S} - \frac{\delta \mathbf{F}}{\delta x}\right)^{\mathrm{R}}_{i+(1/2)}\right] \tag{A4}$$

Conservative schemes can also be derived from the quasi-conservative (1) and the non-conservative (5) form of the equations. These are simpler than (3), but it is necessary to establish the following condition at the discrete level

$$\mathbf{G}_{i+(1/2)} \equiv \left(\mathbf{S}^{\mathrm{c}} - \frac{\delta \mathbf{F}^{\mathrm{c}}}{\delta x}\right)_{i+(1/2)} = \left(\mathbf{S}^{\mathrm{qc}} - \frac{\delta \mathbf{F}^{\mathrm{qc}}}{\delta x}\right)_{i+(1/2)} = \left(\mathbf{S}^{\mathrm{nc}} - \mathbf{J}\,\frac{\delta \mathbf{u}}{\delta x}\right)_{i+(1/2)} \tag{A5}$$

It must be noted that this requires a non-centred discretization of the source term. In the shallow water equations, the equality holds using:

$$\mathbf{S}^{\mathrm{nc}} = \begin{aligned} &\begin{pmatrix} 0 \\ gA(S_0 - S_f - \frac{\delta h}{\delta x} + \frac{1}{B}\frac{\delta A}{\delta x}) \end{pmatrix}_{i+(1/2)}, \quad \mathbf{J}_{i+(1/2)} = \begin{pmatrix} 0 & 1 \\ c^2 - v^2 & 2v \end{pmatrix}_{i+(1/2)} \\ &c_{i+(1/2)} = \sqrt{g\frac{A_{i+(1/2)}}{B_{i+(1/2)}}}, \qquad v_{i+(1/2)} = \frac{Q_{i+1}/\sqrt{A_{i+1}} + Q_i/\sqrt{A_i}}{\sqrt{A_{i+1}} + \sqrt{A_i}} \end{aligned} \tag{A6}$$

If (A5) is satisfied there are three equivalent forms based on the definition of a generalized $\mathbf{G}$: the classical conservative scheme with non-centred source term:

$$\mathbf{G}_{i+(1/2)} \equiv \left(\mathbf{S}^{c} - \frac{\delta \mathbf{F}^{c}}{\delta x}\right)_{i+(1/2)} \tag{A7}$$

the conservative method based on the quasi-conservative formulation

$$\mathbf{G}_{i+(1/2)} \equiv \left(\mathbf{S}^{qc} - \frac{\delta \mathbf{F}^{qc}}{\delta x}\right)_{i+(1/2)} \tag{A8}$$

and the conservative method based on the non-conservative formulation

$$\mathbf{G}_{i+(1/2)} \equiv \left(\mathbf{S}^{nc} - \mathbf{J}\frac{\delta \mathbf{u}}{\delta x}\right)_{i+(1/2)} \tag{A9}$$

all of them (A7)–(A9) requiring a non-centred treatment of the source terms and admitting a wave decomposition in the form (A4):

$$\Delta \mathbf{u}_i^n = \Delta t(\mathbf{G}^{L}_{i-(1/2)} + \mathbf{G}^{R}_{i+(1/2)}) \tag{A10}$$

Conservative schemes can also be derived from the characteristic form of the equations here rewritten as

$$\frac{\partial \mathbf{w}}{\partial t} = \mathbf{P}^{-1}\mathbf{S}^{nc} - \mathbf{\Lambda}\frac{\partial \mathbf{w}}{\partial x} = \mathbf{P}^{-1}\left(\mathbf{S}^{nc} - \mathbf{J}\frac{\partial \mathbf{u}}{\partial x}\right) = \mathbf{P}^{-1}\mathbf{G} \tag{A11}$$

A discrete wave decomposition can be made ensuring conservation

$$(\mathbf{P}^{-1}\mathbf{G})_{i+(1/2)} = (\mathbf{\Omega}^{L}\mathbf{P}^{-1}\mathbf{G})_{i+(1/2)}(\mathbf{\Omega}^{R}\mathbf{P}^{-1}\mathbf{G})_{i+(1/2)} \tag{A12}$$

$$(\mathbf{\Omega}^{L} + \mathbf{\Omega}^{R})_{i+(1/2)} = \mathbf{I} \tag{A13}$$

where $\mathbf{\Omega}^{L}$ and $\mathbf{\Omega}^{R}$ are matrices to be defined in any case. Returning to conservative formulation via $\mathbf{P}$,

$$\Delta \mathbf{u}_i^n = \Delta t[(\mathbf{P}\mathbf{\Omega}^{L}\mathbf{P}^{-1}\mathbf{G})_{i-(1/2)} + (\mathbf{P}\mathbf{\Omega}^{R}\mathbf{P}^{-1}\mathbf{G})_{i+(1/2)}] \tag{A14}$$

A.2. Numerical stability

Another important property is the stability in the propagation of small perturbations. Both the flow equations and the scalar transport differential equations have a linear behaviour in the epropagation of small perturbations. For that reason, the numerical schemes used to discretize are expected not to amplify them. The analysis of the behaviour of the numerical schemes in the case of the linear advection equation is called the Von Neumann analysis [18], and represents the basis of most of the numerical stability conditions.

Given the linear scalar equation:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

and a general three point scheme of the form

$$\begin{aligned} u_i^{n+1} &+ A(\delta u_{i+(1/2)}^{n+1} + \delta u_{i-(1/2)}^{n+1}) + B(\delta u_{i+(1/2)}^{n+1} - \delta u_{i-(1/2)}^{n+1}) \\ &= u_i^n + C(\delta u_{i+(1/2)}^n + \delta u_{i-(1/2)}^n) + D(\delta u_{i+(1/2)}^n - \delta u_{i-(1/2)}^n) \end{aligned} \tag{A15}$$

the amplification factor of the scheme is defined as

$$G = \frac{u_i^{n+1}}{u_i^n} \tag{A16}$$

and a scheme will be stable whenever the following holds for all the Fourier waves in the solution:

$$|G(\phi)| \leqslant 1 \tag{A17}$$

if the above inequality does not hold for some value of ϕ, the perturbations with the corresponding wavelength will be exponentially amplified. The general three point scheme (A15) is stable provided that:

$$\begin{aligned} 2C^2 - D &\leqslant 2A^2 - B \\ 2D^2 - D &\leqslant 2B^2 - B \end{aligned} \tag{A18}$$

A.3. Total variation diminishing

Even though stability ensures that perturbations do not grow in time, they do not prevent oscillations from appearing. The concept of total variation diminishing (TVD) is introduced to define schemes free from numerical oscillations. The total variation is defined as [18]

$$TV^n = \sum_i |\delta u_{i+(1/2)}^n| \tag{A19}$$

The scheme will be said to be TVD if [12]

$$TV^{n+1} \leqslant TV^n \tag{A20}$$

Both numerical oscillations and instabilities increase the total variation of a numerical scheme, therefore the TVD conditions are always more restrictive than the stability conditions. It will next be recalled the conditions ensuring that a general three point scheme is TVD. From (A15):

$$\begin{aligned} \delta u_{i+(1/2)}^{n+1} &+ A(\delta u_{i+(3/2)}^{n+1} - \delta u_{i-(3/2)}^{n+1}) + B(\delta u_{i+(3/2)}^{n+1} - 2\delta u_{i+(1/2)}^{n+1} + \delta u_{i-(1/2)}^{n+1}) \\ &= \delta u_{i+(1/2)}^n + C(\delta u_{i+(3/2)}^n - \delta u_{i-(3/2)}^n) + D(\delta u_{i+(3/2)}^n - 2\delta u_{i+(1/2)}^n + \delta u_{i-(1/2)}^n) \end{aligned}$$

and using triangular inequalities:

$$
\begin{aligned}
&|(1-2B)\delta u_{i+(1/2)}^{n+1}| - |(B+A)\delta u_{i+(3/2)}^{n+1}| - |(B-A)\delta u_{i-(1/2)}^{n+1}| \\
&\quad \leqslant |(1-2D)\delta u_{i+(1/2)}^{n+1}| + |(B+A)\delta u_{i+(3/2)}^{n+1}| + |(B-A)\delta u_{i-(1/2)}^{n+1}|
\end{aligned}
$$

Adding up all terms and rearranging:

$$
\begin{aligned}
&(|1-2B| - |B+A| - |B-A|)\, TV^{n+1} \\
&\quad \leqslant (|1-2D| + |D+C| + |D-C|)\, TV^{n}
\end{aligned}
$$

Therefore the sufficient (but not necessary) condition for a TVD scheme are

$$|1-2B| - |B+A| - |B-A| = |1-2D| + |D+C| + |D-C| = 1$$

which holds whenever:

$$B \leqslant -|A|, \;\; \tfrac{1}{2} \geqslant D \geqslant |C| \tag{A21}$$

This condition also ensures the stability conditions for a three point scheme (A18).

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